Dynamically rational judgment aggregation*

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Abstract

Judgment-aggregation theory has always focused on the attainment of rational collective judgments. But so far, rationality has been understood in static terms: as "coherence" of judgments at a given time, understood as consistency, completeness, and/or deductive closure. By contrast, this paper discusses whether collective judgments can be dynamically rational, so that they change rationally in response to new information. Formally, a judgment aggregation rule is dynamically rational with respect to a given revision operator if, whenever all individuals revise their judgments in light of some information (a learnt proposition), then the new aggregate judgments are the old ones revised in light of this information, i.e., aggregation and revision commute. We prove a general impossibility theorem: if the propositions on the agenda are sufficiently interconnected, no judgment aggregation rule with standard properties is dynamically rational with respect to any revision operator satisfying some mild conditions (familiar from belief revision theory). Our theorem is the dynamic-rationality analogue of some well-known impossibility theorems for static rationality. We also explore how dynamic rationality might be achieved by relaxing some of the conditions on the aggregation rule and/or the revision operator.

1 Introduction

Suppose a group of individuals – say, a committee, expert panel, multi-member court, or other decision-making body – makes collective judgments on some propositions by aggregating its members' individual judgments on those propositions. And now suppose

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the group learns some new information – in the form of the truth of some proposition – that prompts a rational revision of the judgments held. There are two ways in which the group might go about incorporating this new information:

Either (1) the group members individually revise their judgments based on the newly learnt information, and the group then aggregates its members' post-revision judgments.

Or (2) the group first aggregates its members pre-revision judgments and then revises the resulting collective judgments based on the new information.

It would be ideal if both approaches led to the same outcome: (1) revision followed by aggregation, and (2) aggregation followed by revision. If they do, we say that aggregation and revision "commute". In such a case, there is an alignment between "dynamic rationality" at the individual level and its counterpart at the collective level: if the judgments of all individual group members evolve across time in accordance with the given revision method, then so do the group's aggregated judgments. The group functions as a dynamically rational agent through aggregating its members' judgments.

In this paper, we investigate whether we can find reasonable aggregation rules that enable a group to achieve such dynamic rationality: aggregation rules which commute with reasonable revision methods. Surprisingly, this question has not been studied in the judgment-aggregation framework where judgments are binary verdicts on some propositions: "yes"/"no", "true"/"false", "accept"/"reject". (On judgment-aggregation theory, see List and Pettit 2002, Dietrich and List 2007, Nehring and Puppe 2010, Dokow and Holzman 2010a, List and Puppe 2009.) The focus in judgment-aggregation theory has generally been on *static* rationality, namely on whether properties such as consistency, completeness, and deductive closure are preserved when individual judgments are aggregated into collective ones at a given point in time.[1]

By contrast, the question of dynamic rationality has received much attention in the distinct setting of probability aggregation, where judgments aren't binary but take the form of subjective probability assignments to the elements of some algebra. In that context, a mix of possibility and impossibility results has been obtained (e.g., Madansky 1964, Genest 1984, Genest et al. 1986, Dietrich 2010, 2019, Russell et al. 2015). These show that some familiar methods of aggregation – notably, the arithmetic averaging of probabilities – fail to commute with belief revision, understood in broadly

[1]The revision of judgments has been investigated only in a different sense in judgment aggregation theory, namely in peer-disagreement contexts, where revision is prompted not by the learning of some new information but by the fact that others hold distinct judgments. See Pettit (2006) and List (2011).

Bayesian terms, while other methods – particularly geometric averaging – do commute with revision. An investigation of the parallel question in the case of binary judgments is therefore overdue.

Our main result in this paper is, unfortunately, a negative one. We show that, for a large class of familiar judgment aggregation rules, dynamic rationality is unachievable relative to a large class of reasonable judgment revision methods. However, we also show that if we relax some of our main theorem's conditions on the aggregation rule, dynamically rational aggregation becomes logically possible. While some of the identified possibilities are primarily technical and of limited substantive interest, we show that so-called "premise-based" aggregation rules, which are more plausible, are in fact dynamically rational relative to corresponding premise-based revision methods. These are quite special, however, and come at a certain cost, and an open question for future research is whether there might be other reasonable ways to avoid our impossibility result.

Our results reinforce a point that has already been defended in the theory of group agency, namely that it is difficult to achieve rational collective agency merely through the aggregation of individual attitudes and without any *sui generis* deliberative processes at the collective level itself (List and Pettit 2011). Previously, this point has been made primarily in relation to static rationality, where impossibility results have been used to show that rational group attitudes cannot generally supervene on rational individual attitudes in a propositionwise manner. Our results establish a similar point in relation to dynamic rationality. Most of our formal proofs are given in an appendix.

2 The formal setup

We begin with the basic setup from judgment aggregation theory (following List and Pettit 2002 and Dietrich 2007). We assume that there is a set of individuals who hold judgments on some set of propositions, and we are looking for a method of aggregating these judgments into resulting collective judgments. The key elements of this setup are the following:

Individuals. These are represented by a finite and non-empty set N. Its members are labelled 1, 2, ..., n. We assume $n \geq 2$.

Propositions. These are represented in formal logic. For our purposes, a thin notion of "logic" will suffice. Specifically, a *logic*, $\mathbf{L}$, is a non-empty set of formal objects called "propositions", which is endowed with two things:

- a *negation operator*, denoted $\neg$, so that, for every proposition p in $\mathbf{L}$, its negation $\neg p$ is also in $\mathbf{L}$; and
- a well-behaved notion of *consistency*, which specifies, for each set of propositions $S \subseteq \mathbf{L}$, whether S is consistent or inconsistent.[2]

Standard propositional, predicate, modal, and conditional logics all fall under this definition, as do Boolean algebras.[3] We call a proposition p *contradictory* if $\{p\}$ is inconsistent, and *tautological* if $\{\neg p\}$ is inconsistent. Any non-contradictory and non-tautological proposition is called *contingent*.

Agenda. The *agenda* is the set of those propositions from $\mathbf{L}$ on which judgments are to be made. Formally, this is a finite non-empty subset $X \subseteq \mathbf{L}$, which can be partitioned into proposition-negation pairs. Sometimes it is useful to make this partition explicit. We write $\mathcal{Z}$ to denote the set of all proposition-negation pairs in X, each of which is of the form $\{p, \neg p\}$ or abbreviated $\{\pm p\}$. The elements of $\mathcal{Z}$ can be interpreted as the *binary issues* under consideration. Then the agenda X is their disjoint union, formally $X = \bigcup_{Z \in \mathcal{Z}} Z$. Throughout this paper, we assume that double-negations cancel out in agenda propositions.[4] Our focus will be on agendas satisfying a non-triviality condition. To define it, call a set of propositions *minimal inconsistent* if it is inconsistent but all its proper subsets are consistent. Proposition-negation pairs of the form $\{p, \neg p\}$ (with p contingent) are minimal inconsistent, and so are sets of the form $\{p, q, \neg(p \wedge q)\}$, where "$\wedge$" stands for logical conjunction ("and"). We call an agenda *non-simple* if it has at least one minimal inconsistent subset of size greater than two. An example of a non-simple agenda is the set $X = \{\pm p, \pm(p \to q), \pm q\}$, where p might be the proposition "Current atmospheric CO_2 is above 407 ppm", $p \to q$ might be the proposition "If current

[2] *Well-behavedness* is a three-part requirement: (i) any proposition-negation pair $\{p, \neg p\}$ is inconsistent; (ii) any subset of any consistent set is still consistent; and (iii) the empty set is consistent, and any consistent set S has a consistent superset $S' \supseteq S$ which contains a member of every proposition-negation pair $\{p, \neg p\}$.

[3] Readers familiar with probability theory could take $\mathbf{L}$ to be a Boolean algebra on a non-empty set Ω of possible worlds, e.g., $\mathbf{L} = 2^{\Omega}$, with negation defined as set-theoretic complementation and consistency of a set defined as non-empty intersection. The Boolean algebra could also be an abstract rather than set-theoretic Boolean algebra.

[4] To be precise, henceforth, by the *negation* of any proposition $q \in X$ we shall mean the *agenda-internal negation* of q, i.e., the opposite proposition in the binary issue $\{p, \neg p\}$ to which q belongs. This is logically equivalent to the ordinary negation of q and will again be denoted $\neg q$, for simplicity. This convention ensures that $\neg\neg q = q$.

atmospheric CO_2 is above 407 ppm, then the Arctic iceshield will melt by 2050", and q might be the proposition "The Arctic iceshield will melt by 2050". The conditional $p \rightarrow q$ can be formalized in standard propositional logic or in a suitable logic for conditionals. A three-member minimal inconsistent subset of this agenda is $\{p, p \rightarrow q, \neg q\}$.

Judgments. Each individual's (and subsequently the group's) judgments on the given propositions are represented by a *judgment set*, which is a subset $J \subseteq X$, consisting of all those propositions from X that its bearer "accepts" (e.g., affirms or judges to be true). A judgment set J is

- *complete* if it contains a member of each proposition-negation pair from X, i.e., $J \cap Z \neq \emptyset$ for every $Z \in \mathcal{Z}$,
- *consistent* if it is a consistent set in the sense of the given logic, and
- *classically rational* if it has both of these properties.

We write $\mathcal{J}$ to denote the set of *all* classically rational judgment sets on the agenda X. A list of judgment sets $\langle J_1, ..., J_n \rangle$ across the individuals in N is called a *profile (of individual judgment sets)*.

Aggregation rule. A *(judgment) aggregation rule* is a function, F, which maps each profile $\langle J_1, ..., J_n \rangle$ in some domain $\mathcal{D}$ of admissible profiles (often $\mathcal{D} = \mathcal{J}^n$) to a collective judgment set $J = F(J_1, ..., J_n)$. A standard example is *majority rule*, which is defined as follows: for each $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$,

$$F(J_1, ..., J_n) = \{p \in X : |\{i : p \in J_i\}| > \frac{n}{2}\}.$$

A typical research question in judgment aggregation theory is whether we can find aggregation rules that satisfy certain requirements of democratic responsiveness to the individual judgments and collective rationality. Usually, the focus is on the attainment of *static* rationality at the collective level, i.e., rationality of the collective judgments at a particular point in time, especially their consistency and perhaps their completeness. Here, by contrast, our focus will be on requirements of *dynamic* rationality. To introduce these, we must first introduce the notion of judgment revision.

3 Judgment revision

The idea we wish to capture is that whenever any individual (or subsequently the group) learns some new information, in the form of the truth of some proposition, this individual (or the group) must incorporate the learnt information in the judgments held – an idea familiar from belief revision theory in the tradition of Alchourrón, Gärdenfors, and Makinson (1985) (see also Rott 2001). Our central concept is that of a *judgment revision operator*. This is a function which assigns to any pair (J, p) of an initial judgment set $J \subseteq X$ and a learnt proposition $p \in X$ a new judgment set $J|p$. We can interpret this as the revised judgment set, given p. It is convenient not to restrict the domain of admissible inputs and outputs of a revision operator, so that it can take any *logically possible* pair (J, p) as input, with $J \subseteq X$ and $p \in X$, and produce any subset of X as output. Formally, it is a function from $2^X \times X$ into 2^X.

We call a revision operator *regular* if it satisfies the following two minimal conditions:

(i) it is *successful*, i.e., $p \in J|p$ for any pair (J, p), and

(ii) it is *conservative*, i.e., $J|p = J$ for any pair (J, p) such that $p \in J$.

Condition (i) ensures that any learnt proposition p is indeed incorporated in the post-revision judgment set ("accept what you learn"). Condition (ii) ensures that if the learnt proposition is already accepted, then nothing changes ("no news, no change"). We further call a revision operator *rationality-preserving* if whenever $J \in \mathcal{J}$, we have $J|p \in \mathcal{J}$ for all non-contradictory propositions $p \in X$.

These definitions are well-illustrated by the class of *distance-based revision operators*, familiar from belief revision theory. Such operators require that when a judgment set is revised in light of some new information, the post-revision judgments remain as "close" as possible to the pre-revision judgments, subject to the constraint that the learnt information be incorporated and no inconsistencies be introduced. Different distance-based operators spell out the notion of "closeness" in different ways.

To make this precise, we first consider a *distance metric* on judgment sets (such metrics have been introduced in the area of judgment aggregation by Konieczny and Pino Pérez 2002 and Pigozzi 2006). This is a function d that assigns to any pair of judgment sets $J, J' \subseteq X$ a non-negative real number $d(J, J')$ interpreted as the "distance" between J and J', subject to the minimal condition that $d(J, J') = 0$ if and only if $J = J'$. A simple example of a distance metric is the *Hamming distance*, according to which $d(J, J')$ is the number of propositions in X on which J and J' disagree, i.e.,

$$d(J, J') = \left| \left\{ p \in X : p \in J \not\Leftrightarrow p \in J' \right\} \right|.$$

Now, given a distance metric d, we can define a corresponding judgment revision operator. For any (J, p), let $J|p$ be a judgment set J' satisfying the following constraints:

- J' contains p,
- J' is classically rational, except possibly when J is not classically rational or p is contradictory,
- J' has minimal distance from J among those judgment sets satisfying the first two constraints.[5]

By construction, any distance-based revision operator is successful (because of the first bullet point), rationality-preserving (because of the second), and conservative (because of the last bullet point, given our minimal condition on a distance metric). We will later construct several other revision operators, but for the moment, the present example should suffice as an illustration.

4 Can aggregation and revision commute?

We are now ready to turn to this paper's question. As noted, we would ideally want any decision-making group to employ a judgment aggregation rule and a revision operator that generate the same collective judgments irrespective of whether revision takes place before or after aggregation. This requirement (an analogue of the classic "external Bayesianity" condition in probability aggregation theory, as in Madansky 1964, Genest 1984, and Genest et al. 1986) is captured by the following condition on the aggregation rule F and the revision operator $|$:

Dynamic rationality. For any profile $\langle J_1, ..., J_n \rangle$ in the domain of F and any learnt proposition $p \in X$ where the revised profile $\langle J_1|p, ..., J_n|p \rangle$ is also in the domain of F, $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$.

To see that this condition is surprisingly hard to satisfy, consider an example. Suppose a three-member group is making judgments on the agenda $X = \{\pm p, \pm(p \to q), \pm q\}$, where $p \to q$ is understood as a subjunctive conditional. That is, apart from the subsets of X that include a proposition-negation pair, the only inconsistent subset of X is $\{p, (p \to q), \neg q\}$.[6] Suppose, further, the group members' initial judgments are as shown on the left-hand side Table 1, where "yes" stands for the acceptance of a proposition and "no" for the acceptance of its negation.

[5] Insofar as there need not be a unique such distance-minimizing J', the choice of J' may require a tie-breaking criterion.

[6] This subjunctive understanding of $p \to q$ contrasts with the material one, where $p \to q$ is understood less realistically as $\neg p \vee q$. On the material understanding, the subsets $\{p, \neg(p \to q), q\}$, $\{\neg p, \neg(p \to q), q\}$, and $\{\neg p, \neg(p \to q), \neg q\}$ would also be deemed inconsistent.

Table 1: A simple example

	Before learning p			After learning p		
	p	$p \to q$	q	p	$p \to q$	q
Individual 1	No	No	Yes	Yes	No	Yes
Individual 2	No	Yes	No	Yes	Yes	Yes
Individual 3	No	No	No	Yes	No	No
Majority	No	No	No	Yes	No	Yes

Suppose now that the aggegation rule is majority rule and the revision operator is based on the Hamming distance, with some tie-breaking provision such that, in the case of a tie, one is more ready to change one's judgment on p or q than on $p \to q$. If the individuals learn the truth of p and revise their judgments, they arrive at the post-revision judgments shown on the right-hand side of Table 1. Aggregating those judgments yields the collective judgment set $\{p, \neg(p \to q), q\}$. By contrast, if the individuals first aggregate their pre-revision judgments, they arrive at the majority judgment set $\{\neg p, \neg(p \to q), \neg q\}$, and its revision in response to learning p yields the judgment set $\{p, \neg(p \to q), \neg q\}$. Thus the group arrives at a different collective judgment set depending on whether aggregation precedes revision or the other way round: the combination of majority rule and distance-based revision is not dynamically rational.

At first sight, one might think that this problem is just an artifact of majority rule or our specific distance-based revision operator, or that it is somehow unique to our example. However, our first formal result shows that the problem is more general. Define a *uniform quota rule*, with acceptance threshold $m \in \{1, 2, ..., n\}$, as the aggregation rule with domain $\mathcal{J}^n$ such that, for each $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$,

$$F(J_1, ..., J_n) = \{p \in X : |\{i : p \in J_i\}| \geq m\}.$$

Majority rule is a special case of a uniform quota rule, namely the one where m is the smallest integer greater than $\frac{n}{2}$. We have:

Theorem 1. If the agenda X is non-simple, then no uniform quota rule whose threshold is not the unanimity threshold n is dynamically rational with respect to any regular rationality-preserving revision operator.

In short, replacing majority rule with some other uniform quota rule with threshold less than n wouldn't solve our problem of dynamic irrationality, and neither would replacing our distance-based revision operator with some other regular rationality-preserving revision operator. In fact, the problem generalizes further, as shown in the next section.

5 A general impossibility theorem

We will now abstract away from the details of any particular aggregation rule, and suppose instead we are looking for an aggregation rule F that satisfies the following general conditions:

Universal domain: The domain of admissible inputs to the aggregation rule F is the set of all classically rational profiles, i.e., $\mathcal{D} = \mathcal{J}^n$.

Non-imposition: F does not always deliver the same antecedently fixed output judgment set J, irrespective of the individual inputs, i.e., F is not a constant function.

Monotonicity: Additional individual support for an accepted proposition does not overturn the proposition's acceptance, i.e., for any profile $\langle J_1, ..., J_n \rangle \in \mathcal{D}$ and any proposition $p \in F(J_1, ..., J_n)$, if any J_i not containing p is replaced by some J_i' containing p and the modified profile $\langle J_1, ..., J_i', ..., J_n \rangle$ remains in $\mathcal{D}$, then $p \in F(J_1, ..., J_i', ..., J_n)$.

Non-oligarchy: There is no non-empty set of individuals $M \subseteq N$ (a set of "oligarchs") such that, for every profile $\langle J_1, ..., J_n \rangle \in \mathcal{D}$, $F(J_1, ..., J_n) = \cap_{i \in M} J_i$.

Systematicity: The collective judgment on each proposition is determined fully and neutrally by individual judgments on that proposition. Formally, for any propositions $p, p' \in X$ and any profiles $\langle J_1, ..., J_n \rangle, \langle J_1', ..., J_n' \rangle \in \mathcal{D}$, if, for all $i \in N$, $p \in J_i \Leftrightarrow p' \in J_i'$, then $p \in J \Leftrightarrow p' \in J'$, where $J = F(J_1, ..., J_n)$ and $J' = F(J_1', ..., J_n')$.

Why are these conditions initially plausible? The reason is that, for each of them, a violation would entail a cost. Violating universal domain would mean that the aggregation rule is not fully robust to pluralism in its inputs; it would be undefined for some classically rational judgment profiles. Violating non-imposition would mean that the collective judgments are totally unresponsive to the individual judgments, which is completely undemocratic. Violating monotonicity could make the aggregation rule erratic in some respect: an individual could come to accept a particular collectively accepted proposition and thereby overturn its acceptance. Violating non-oligarchy would mean two things. First, the collective judgments would depend only on the judgments of the "oligarchs", which is undemocratic when $M \neq N$; and second, the collective judgments would be incomplete with respect to any binary issue on which there is the slightest disagreement among the oligarchs, which would lead to widespread indecision, except when M is singleton. Important special cases of oligarchic rules are *dictatorships of one individual* (where M is singleton) and *unanimity rule* (where $M = N$). Violating systemacity, finally, would mean that the collective judgment on each proposition is no longer determined as a proposition-independent function of individual judgments on that proposition. It may then either depend on individual judgments on other propositions too (a lack of *propositionwise independence*), or the pattern of dependence may vary from proposition to proposition (a lack of *neutrality*). Systematicity – the conjunction of propositionwise independence and neutrality – is the most controversial condition among the five. But it's worth noting that it is satisfied by majority rule and all uniform quota rules. Indeed, majority rule and uniform quota rules (except the unanimity rule) satisfy all five conditions.

Our main theorem shows that, for non-simple agendas, the present five conditions are incompatible with dynamic rationality:

Theorem 2. If the agenda X is non-simple, then no aggregation rule satisfying universal domain, non-imposition, monotonicity, non-oligarchy, and systematicity is dynamically rational with respect to any regular rationality-preserving revision operator.

So, the problem identified by Theorem 1 is not restricted to uniform quota rules, but extends to all aggregation rules satisfying our conditions. Moreover, since practically all non-trivial agendas are non-simple, the impossibility applies very widely. In the next section, we show that all of the theorem's conditions – not only the ones on the aggregation rule but also the one on the agenda – are needed for the present impossibility result, i.e., the impossibility ceases to hold if any one of these conditions is dropped.

6 Non-redundancy of the conditions

We will first run through the five conditions on the aggregation rule and show that, for each of them, there exist aggregation rules on some non-simple agendas which satisfy all of our conditions except the given one, while being dynamically rational with respect to some regular rationality-preserving revision operators. Importantly, many of these examples are relatively contrived and thus more of theoretical rather than practical interest. We will then show that the theorem's condition on the agenda – non-simplicity – is needed for the impossibility too. Thus we could amend the theorem's antecedent clause by writing "If, and only if, the agenda X is non-simple".

6.1 Possibilities without universal domain

To see that Theorem 2 would fail to hold without the condition of universal domain, we take any non-simple agenda X and construct two non-trivial examples of restricted domains $\mathcal{D} \subseteq \mathcal{J}^n$ on which majority rule is dynamically rational with respect to some regular rationality-preserving revision operator. Since majority rule clearly satisfies the rest of our conditions (non-imposition, monotonicity, non-oligarchy, and systematicity), the examples establish our point.

For our first example, we define the domain as follows:

$$\mathcal{D} = \left\{ \langle J_1, ..., J_n \rangle \in \mathcal{J}^n : |\{i \in N : J_i = J\}| > \frac{n}{2} \text{ for some } J \in \mathcal{J} \right\},$$

i.e., $\mathcal{D}$ consists of all rational judgment profiles in which a majority of individuals hold the same judgment set. It is easy to verify that, whenever a profile $\langle J_1, ..., J_n \rangle$ is in $\mathcal{D}$, then the revised profile $\langle J_1|p, ..., J_n|p \rangle$ is still in $\mathcal{D}$, for any non-contradictory proposition $p \in X$ and any regular rationality-preserving revision operator. Moreover, if F is majority rule on $\mathcal{D}$, then $F(J_1, ..., J_n)$ is simply the judgment set J held by a majority of individuals in $\langle J_1, ..., J_n \rangle$, so that $F(J_1, ..., J_n)|p = J|p$. The revised profile $\langle J_1|p, ..., J_n|p \rangle$ has the property that the majority of individuals who previously held the judgment set J come to hold the judgment set $J|p$, so that the latter is also the majority outcome. Hence $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$, as required.

Our second example invokes the idea that the propositions in X can be ordered from "left" to "right" on some cognitive or ideological dimension, in such a way that all individuals' judgments are structured by that order. Specifically, consider a linear order $\leq$ on X, where, for any two propositions $p, q \in X$, $p \leq q$ means that "p is (weakly) to the left of q". We call a profile $\langle J_1, ..., J_n \rangle$ *single-plateaued* relative to $\leq$ if, for every individual $i \in N$,

$$J_i = \{p \in X : p_{left} \leq p \leq p_{right}\} \text{ for some } p_{\text{left}}, p_{\text{right}} \in X,$$

i.e., the individual's judgment set forms a connected interval (a "plateau" of accepted propositions) with respect to $\leq$, ranging from p_{left} to p_{right}. It is already known that single-plateauedness, combined with individual-level consistency, is sufficient for consistent majority judgments (Dietrich and List 2010). To explain how single-plateauedness can also help with dynamic rationality, let $\mathcal{J}_{\leq}$ denote the subset of $\mathcal{J}$ consisting of all classically rational judgment sets that are single-plateaued relative to $\leq$. Define a judgment revision operator as follows: for any pair (J, p),

- if $J \in \mathcal{J}_{\leq}$ and $\mathcal{J}_{\leq}$ contains at least one judgment set containing p, let $J|p$ be the (unique) judgment set $J' \in \mathcal{J}_{\leq}$ containing p whose Hamming distance from J is minimal (so that judgment revision simply shifts the plateau of accepted propositions minimally until it contains p while remaining classically rational);
- otherwise, let $J|p$ be any judgment set $J' \subseteq X$ containing p whose Hamming distance from J is minimal, subject to the constraint that if $J \in \mathcal{J}$ and p is non-contradictory, then $J' \in \mathcal{J}$.

One can now show that, on the domain $\mathcal{D} = \mathcal{J}^n_{\leq}$, majority rule (in a group N with odd-numbered size n) is dynamically rational with respect to the revision operator just defined. The reason is that whenever a classically rational profile $\langle J_1, ..., J_n \rangle$ is single-plateaued relative to $\leq$, the majority judgments will coincide with the individual judgments of a particular profile-specific individual (technically, the median individual relative to some left-right order of the individuals that can be suitably constructed for the given profile), and even if all individuals revise their judgments based on learning a proposition p in line with the first bullet point, the majority judgments will still coincide with the revised judgments of that same individual. Details are given in the appendix.

6.2 Possibilities without non-imposition

To see that Theorem 2 would fail to hold without the condition of non-imposition, we take any non-simple agenda X and any regular rationality-preserving revision operator, and note that the following, rather absurd aggregation rule is dynamically rational while satisfying the rest of our conditions: for any profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$,

$$F(J_1, ..., J_n) = X,$$

i.e., the collective judgment set is always identical to the agenda in its entirety. Of course, this aggregation rule is completely unresponsive to the individual judgments and produces totally inconsistent collective judgments. Nonetheless, it satisfies universal domain, monotonicity, non-oligarchy, and systematicity, while also satisfying dynamic rationality. (Note that, for any regular revision operator and any proposition p, $X|p = X$.) Let's call this aggregation rule the *absurd rule*.

One might wonder whether there are any less absurd examples of dynamically rational aggregation rules when we drop non-imposition. In fact, there are none. Our proof of Theorem 2 shows that, for any non-simple agenda X and any regular rationality-preserving revision operator, the absurd rule is the unique dynamically rational aggregation rule satisfying the rest of our conditions. Thus Theorem 2 would continue to hold if we were to replace non-imposition with the requirement that the aggregation rule should not be the absurd rule.

6.3 Possibilities without monotonicity

To see that Theorem 2 would fail to hold without the condition of monotonicity, we show that, for some non-simple agendas, one can construct non-monotonic aggregation rules that are dynamically rational with respect to some regular rationality-preserving revision operator, while satisfying the rest of our conditions. Specifically, we consider a non-simple agenda X with the following properties:

- X is *affine*, in the sense that every minimal inconsistent subset $Y \subseteq X$ remains inconsistent after negating any two (or any even number) of its members.[7]

- For each contingent proposition $p \in X$, there exists a subagenda X_p (a non-empty subset of X closed under negation) which contains p and shares an even number of propositions with any minimal inconsistent subset of $Y \subseteq X$, i.e., $|Y \cap X_p| \in \{0, 2, 4, 6, ...\}$.

An example of such an agenda is $X = \{\pm p, \pm q, \pm(p \leftrightarrow q)\}$, where p and q are logically independent and $\leftrightarrow$ is the material biconditional. This agenda is clearly non-simple. To see that it is affine, note that its minimal inconsistent subsets, besides all proposition-negation pairs, are $\{\neg p, q, p \leftrightarrow q\}$, $\{p, \neg q, p \leftrightarrow q\}$, $\{p, q, \neg(p \leftrightarrow q)\}$, $\{\neg p, \neg q, \neg(p \leftrightarrow q)\}$. Negating any two members of any one of these sets yields another one of them. Furthermore, for each $p \in X$, we can take X_p to be any subagenda of X that includes $\{\pm p\}$ and exactly one other proposition-negation pair. Then X_p shares an even number of propositions with any minimal inconsistent subset of X.

Let us now define a judgment revision operator as follows: for any pair (J, p), let

$$J|p = \begin{cases} J & \text{if } p \in J, \\ (X_p \backslash J) \cup (J \backslash X_p) & \text{if } p \notin J \text{ and } J \in \mathcal{J}, \\ \text{any judgment set containing } p & \text{if } p \notin J \text{ and } J \notin \mathcal{J}, \end{cases}$$

where X_p is the above-defined subagenda if p is contingent and is $\{\pm p\}$ if p is non-contingent. Although this revision operator is admittedly a bit contrived, one can verify that it is both regular and rationality-preserving (details are in the appendix). It now turns out that an even more contrived kind of aggregation rule – a so-called *parity rule* (as introduced by Dokow and Holzman 2010a) – is dynamically rational with respect to this revision operator, while satisfying all of our conditions except monotonicity. To define it, let M be any odd-sized subset of N, and for any profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$, let

$$F(J_1, ..., J_n) = \{p \in X : |\{i \in M : p \in J_i\}| \text{ is odd}\},$$

[7] The negation of *affiness* is *non-affineness* or *pair-negatability*, which is the condition that X has at least one minimal inconsistent subset Y in which we can find two (or an even number of) distinct propositions whose negation renders Y consistent. The name "affineness" is due to Dokow and Holzman (2010a), who introduced this condition in an explicitly algebraic form.

i.e., the set of collectively accepted propositions consists of all propositions that are accepted precisely by an odd number of individuals in M. Clearly, this aggregation rule is non-monotonic. However, we show in the appendix that, for any agenda of the specified kind – such as $X = \{\pm p, \pm q, \pm(p \leftrightarrow q)\}$ – the present aggregation rule is dynamically rational with respect to the revision operator just defined. Furthermore, a parity rule satisfies universal domain, non-imposition, systematicity, and non-oligarchy (assuming $|M| \geq 3$). To be sure, this possibility is of no substantive interest and only illustrates the mathematical point that the monotonicity condition is needed in Theorem 2.

6.4 Possibilities without non-oligarchy

To see that Theorem 2 would fail to hold without the condition of non-oligarchy, we give two examples of oligarchic aggregation rules that are dynamically rational with respect to some (or even any) regular rationality-preserving revision operator for some (or even any) non-simple agenda. These examples suffice to illustrate the non-redundancy of the non-oligarchy condition in our theorem because oligarchic rules always satisfy universal domain, non-imposition, monotonicity, and systematicity. Recall that an oligarchy (as discussed by Gärdenfors 2006, Dietrich and List 2008, and Dokow and Holzman 2010b) is defined by fixing some non-empty set $M \subseteq N$ of individuals (the "oligarchs") such that, for every profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$, we have

$$F(J_1, ..., J_n) = \cap_{i \in M} J_i.$$

Our first example is a trivial one, namely a dictatorship of one individual; here the set of oligarchs is singleton, i.e., $M = \{i\}$ for some fixed $i \in N$. Clearly, if we have $F(J_1, ..., J_n) = J_i$ for every profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$, then it trivially follows that $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$, irrespective of the agenda X and the revision operator.

For a less trivial example, which permits more than one oligarch, consider an agenda of the form $X = \{\pm p_1, \pm p_2, \pm p_3\}$, where the set of all classically rational judgment sets is

$$\mathcal{J} = \{\{p_1, p_2, p_3\}, \{\neg p_1, \neg p_2, p_3\}, \{p_1, \neg p_2, \neg p_3\}, \{\neg p_1, p_2, \neg p_3\}\},$$

i.e., the set of all complete subsets of X in which an even number of propositions (either zero or two) is negated. Such an agenda is non-simple: a minimal inconsistent subset of X of size three is $\{\neg p_1, \neg p_2, \neg p_3\}$. An example is once again $X = \{\pm p, \pm q, \pm(p \leftrightarrow q)\}$, where p and q are logically independent and $\leftrightarrow$ is the material biconditional. Now we first define a regular rationality-preserving judgment revision operator for X, and we then show that *any* oligarchic aggregation-rule is dynamically rational with respect to it.

To construct the desired revision operator, we start from an assignment of a revised judgment set $J_p \in \mathcal{J}$ for every pair (J, p), where $J \in \mathcal{J}$ and $p \in X$. We construct this assignment such that

- for any $p \in X$ and any $J \in \mathcal{J}$, if $p \in J$, then $J_p = J$, and
- for any $p \in X$ and any $J, J' \in \mathcal{J}$, if J and J' are distinct and do not not contain p (i.e., they are the two distinct judgment sets in $\mathcal{J}$ containing $\neg p$), then J_p and J'_p are distinct and contain p (i.e., they are the two distinct judgment sets in $\mathcal{J}$ containing p).

These properties jointly imply that $J_p \in \mathcal{J}$ and $p \in J_p$. We can think of the assignment of a judgment set J_p to each pair (J, p) as the restriction of the desired judgment revision operator to the domain $\mathcal{J} \times X$. Our goal is to extend this operator to *all* pairs (J, p) with $J \subseteq X$ and $p \in X$.

For the purposes of our example, we fully define the revision operator for all pairs (J, p) where J belongs to the set $\mathcal{J}^+$ of all consistent and deductively closed subsets of X (a superset of $\mathcal{J}$). For all other pairs, the operator can be defined arbitrarily, subject only to the restrictions of regularity (i.e., $p \in J|p$, and if $p \in J$ then $J|p = J$). Now, for any pair (J, p) with $J \in \mathcal{J}^+$ and $p \in X$, we define

$$J|p = \bigcap_{J' \in \mathcal{J} : J \subseteq J'} J'_p,$$

i.e., $J|p$ is the intersection of all revised judgment sets of the form J'_p, where J' is a complete and consistent extension of J. To give an intuition for this definition, note that any consistent and deductively closed judgment set J can be expressed as the intersection of all its complete and consistent extensions $J' \supseteq J$. So, our definition says that J is revised by revising all its complete and consistent extensions and taking the intersection of the revised judgment sets. As a special case of this, we have $J|p = J_p$ whenever J is complete and consistent.

This completes the definition of our revision operator. Note that this operator is rationality-preserving and regular. It is rationality-preserving because, for any $J \in \mathcal{J}$, we have $J|p = J_p$, and the latter is in $\mathcal{J}$. It is successful – the first part of regularity – because, for any $J \in \mathcal{J}^+$ and any $p \in X$, the revised judgment set $J|p$ is the intersection of sets of the form J'_p, which each contain p, so that $J|p$ also contains p. Moreover, for any $J \notin \mathcal{J}^+$, $J|p$ contains p by stipulation.

The operator is conservative – the second part of regularity – because, for any $J \in \mathcal{J}^+$

and any $p \in J$, we have

$$J|p = \bigcap_{J' \in \mathcal{J} : J \subseteq J'} J'_p = \bigcap_{J' \in \mathcal{J} : J \subseteq J'} J' = J.$$

The first identity holds by the definition of the revision operator. The second identity holds because we have $J'_p = J'$ whenever $p \in J'$. The third identity holds because J, being consistent and deductively closed, is identical to the intersection of all its complete and consistent extensions. Once again, for any $J \notin \mathcal{J}^+$, conservativeness (if $p \in J$ then $J|p = J$) holds by stipulation.

In the appendix we prove that, on the given non-simple agenda X, every oligarchic aggregation rule is dynamically rational with respect to the constructed revision operator. Let F be any oligarchic aggregation rule with the set $M \subseteq N$ of oligarchs. Our proof establishes that, for every $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ and every $p \in X$,

$$F(J_1, ..., J_n)|p = F(J_1|p, ..., J_n|p),$$

i.e.,

$$\left(\cap_{i \in M} J_i\right)|p = \cap_{i \in M}(J_i|p). \quad (1)$$

To give an intuition for this result, we briefly explain why the judgment set on the left-hand side of identity (1) is included in the judgment set on the right-hand side. The converse inclusion is harder to show, and we refer the reader to the appendix for the full proof. We begin by noting that $\cap_{i \in M} J_i$ is consistent and deductively closed, being the intersection of several consistent and complete judgment sets. Therefore, the definition of our revision operator allows us to rewrite the judgment set on the left-hand side of identity (1) as follows:

$$\left(\cap_{i \in M} J_i\right)|p = \bigcap_{J' \in \mathcal{J} : (\cap_{i \in M} J_i) \subseteq J'} J'_p.$$

Further, the judgment set on the right-hand side of identity (1), $\cap_{i \in M}(J_i|p)$, can be re-expressed as $\cap_{i \in M}(J_i)_p$, since each $J_i \in \mathcal{J}$, and thus as

$$\bigcap_{J' \in \{J_i : i \in M\}} J'_p. \quad (2)$$

Since each J_i is a complete and consistent extension of the intersection $\cap_{i \in M} J_i$, expression (2) includes

$$\bigcap_{J' \in \mathcal{J} : (\cap_{i \in M} J_i) \subseteq J'} J'_p, \quad (3)$$

because expression (3) is simply an intersection of more sets than expression (2): the sets being intersected in (3) include all those being intersected in (2). This establishes that

$$(\cap_{i \in M} J_i)\,|p \subseteq \cap_{i \in M}(J_i|p),$$

as desired. As noted, in the appendix, we show that the two judgment sets are in fact identical.

6.5 Possibilities without systematicity

To see that Theorem 2 would fail to hold without the condition of systematicity, we show that, for some non-simple agendas, one can construct aggregation rules which satisfy all of our conditions except systematicity and are dynamically rational with respect to some regular rationality-preserving revision operator. Specifically, we consider an agenda X of the form $X = \{\pm p : p \in Y\}$, where Y is the only minimal inconsistent subset of X apart from the proposition negation pairs $\{p, \neg p\} \subseteq X$ and where Y has three or more elements. An example is the earlier agenda $X = \{\pm p, \pm(p \rightarrow q), \pm q\}$, where $p \rightarrow q$ is a subjunctive conditional, so that the only minimal inconsistent subsets of X are the proposition-negation pairs and $\{p, (p \rightarrow q), \neg q\}$. Now we define a revision operator as follows:

$$J|p = \begin{cases} J & \text{if } p \in J, \\ \{p\} \cup (J \backslash \{\neg p\}) & \text{if } p \notin J \text{ and } p \notin Y, \\ \{p\} \cup \{\neg q : q \in Y \backslash \{p\}\} & \text{if } p \notin J \text{ and } p \in Y. \end{cases}$$

It is easy to see that this revision operator is regular. To see that it is also rationality-preserving, take any $J \in \mathcal{J}$ and any $p \in X$. (In the present agenda, all propositions are non-contradictory.) The revised judgment set $J|p$ is complete because J itself is complete and Y contains a member of every proposition-negation pair $\{p, \neg p\} \subseteq X$. Furthermore, $J|p$ is consistent because it includes neither Y itself nor any proposition-negation pair $\{p, \neg p\} \subseteq X$, and so it includes no minimal inconsistent set.

We now show that the following aggregation rule is dynamically rational with respect to this revision operator. For any profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$, let $F(J_1, ..., J_n)$ consist of

- all $p \in Y$ such that every J_i contains p, and
- all $p \in X \backslash Y$ such that at least one J_i contains p.

We can think of this aggregation rule as an asymmetric unanimity rule. Propositions in Y are collectively accepted if and only if they are unanimously accepted, while propositions outside Y are collectively accepted if and only if they are not unanimously rejected.

It is evident that this aggregation rule satisfies universal domain, non-imposition, monotonicity, and non-oligarchy. It is also evident that it violates systematicity: the collective acceptance criterion is not the same for all propositions (a lack of neutrality). To see that it is dynamically rational with respect to the constructed revision operator, we distinguish between three cases.

- *Case 1*: $p \in Y$ and $p \in J_i$ for all $i \in N$. Then $p \in F(J_1, ..., J_n)$. Because the revision operator is conservative, $J_i|p = J_i$ for every $i \in N$ and $F(J_1, ..., J_n)|p = F(J_1, ..., J_n)$. So, $F(J_1, ..., J_n)|p = F(J_1|p, ..., J_n|p)$.

- *Case 2*: $p \in Y$ and $p \notin J_i$ for some $i \in N$. Then $J_i|p = \{p\} \cup \{\neg q : q \in Y \setminus \{p\}\}$. This means that, in the profile $\langle J_1|p, ..., J_n|p \rangle$, p is unanimously accepted (because the revision operator is conservative), while all propositions outside Y (namely, those in $\{\neg q : q \in Y \setminus \{p\}\}$) are accepted by at least one individual (namely, individual i). So, $F(J_1|p, ..., J_n|p) = \{p\} \cup \{\neg q : q \in Y \setminus \{p\}\}$. Meanwhile, since $p \in Y$ and p is not unanimously accepted in the profile $\langle J_1, ..., J_n \rangle$, we have $p \notin F(J_1, ..., J_n)$, and so $F(J_1, ..., J_n)|p = \{p\} \cup \{\neg q : q \in Y \setminus \{p\}\}$. This shows that $F(J_1, ..., J_n)|p = F(J_1|p, ..., J_n|p)$.

- *Case 3*: $p \notin Y$. Here, revision of any judgment set simply leads to the acceptance of p and the non-acceptance of $\neg p$, while nothing else changes. So, the profile $\langle J_1|p, ..., J_n|p \rangle$ displays unanimous acceptance of p and coincides with the profile $\langle J_1, ..., J_n \rangle$ on all proposition-negation pairs distinct from $\{p, \neg p\}$. Then $F(J_1|p, ..., J_n|p)$ contains p and coincides with $F(J_1, ..., J_n)$ on all other proposition-negation pairs. Furthermore, $F(J_1, ..., J_n)|p$ also contains p and concides with $F(J_1, ..., J_n)$ on all other proposition-negation pairs. Hence, $F(J_1, ..., J_n)|p = F(J_1|p, ..., J_n|p)$.

At the end of the paper, we consider another class of aggregation rules violating systematicity which offer an escape route from our impossibility result, though that route requires relaxing some of our conditions on the revision operator too.

6.6 Possibilities for simple agenads

We have seen that all of Theorem 2's conditions on the aggregation rule are needed for the impossibility result. We now turn to the theorem's condition on the agenda. Recall that the theorem asserts that the impossibility arises if the agenda is non-simple: it has at least one minimal inconsistent subset with more than two propositions. We will show that, if the agenda is simple, there exist aggregation rules that satisfy the required conditions while being dynamically rational with respect to a natural kind of revision operator.

Consider any simple agenda X. Let the revision operator be as follows. For any $J \subseteq X$ and any $p \in X$,

- if $J \in \mathcal{J}$, then $J|p = \{q_p : q \in J\}$, where

$$q_p = \begin{cases} q & \text{if } \{q, p\} \text{ is consistent,} \\ \neg q & \text{otherwise;} \end{cases}$$

- if $J \notin \mathcal{J}$, then $J|p$ can be defined arbitrarily, subject to the regularity conditions that (i) $p \in \mathcal{J}|p$ and (ii) if $p \in J$, then $J|p = J$.

By definition, this operator is regular, and since X is simple, it can also be seen to be rationality-preserving. Moreover, the operator has the special feature of being *local*: the revised judgment on any proposition $q \in X$ depends only on the initial judgment on q and on the learnt proposition p.[8] The following result holds:

Proposition 1. If the agenda X is simple, then every aggregation rule satisfying universal domain, collective rationality, propositionwise independence (or systematicity), and unanimity preservation is dynamically rational with respect to the revision operator just defined.

Here *universal domain* is as before; *collective rationality* is the requirement that $F(J_1, ..., J_n) \in \mathcal{J}$ for every profile $\langle J_1, ..., J_n \rangle \in \mathcal{D}$; *propositionwise independence* is a weakened version of systematicity, where the quantification is restricted to pairs of propositions p, p' with $p = p'$; and *unanimity preservation* is the requirement that $F(J, ..., J) = J$ for every unanimous profile $\langle J, ..., J \rangle \in \mathcal{D}$. Unanimity preservation strengthens non-imposition. An example of an aggregation rule satisfying all of these conditions (if X is simple and n is odd) is majority rule, which of course also satisfies all of the conditions of Theorem 2.

This shows that non-simplicity of the agenda is not only sufficient for our impossibility result, but also necessary. In fact, this is true not just in the case of Theorem 2, but also in the case of Theorem 1.

[8] The present definition has the interesting implication that, for a simple agenda X, if the initial judgment set J is classically rational and the learnt proposition p is non-contradictory, then $J|p$ is the unique classically rational judgment set that minimizes the Hamming distance from J subject to containing p. For general agendas, revision by minimizing Hamming distance is neither unique, nor local.

7 Premise-based revision and aggregation

The possibilities of dynamically rational aggregation we have considered in response to Theorem 2 have mostly served to prove the technical point that none of the theorem's conditions is redundant. We conclude by considering a possibly more interesting escape route from our impossibility result, which works for non-simple agendas and involves relaxing not only some of the conditions on the aggregation rule but also some of those on the revision operator. Specifically, we will show that so-called *premise-based aggregation rules* – the best-known aggregation rules giving up systematicity – are dynamically rational if revision is defined in a corresponding premise-based way. The cost of this, however, is a relaxation of our regularity conditions on revision itself.

Let us begin by introducing the idea of premise-based aggregation (for earlier definitions related to the present one, see List and Pettit 2002, Dietrich 2006, and Dietrich and Mongin 2009; for other discussions, see Kornhauser and Sager 1986, Pettit 2001, Chapman 2002, and Bovens and Rabinowicz 2006). Suppose the agenda X can be partitioned into a subagenda of premises and a subagenda of conclusions. Formally, we represent this partition by partitioning the set $\mathcal{Z}$ of binary issues into a set $\mathcal{Z}_{\text{prem}}$ of "premise issues" and a set $\mathcal{Z}_{\text{conc}}$ of "conclusion issues". Then the subagendas of premises and conclusions are $X_{\text{prem}} = \bigcup_{Z \in \mathcal{Z}_{\text{prem}}} Z$ and $X_{\text{conc}} = \bigcup_{Z \in \mathcal{Z}_{\text{conc}}} Z$. As an illustration, consider again the agenda $X = \{\pm p, \pm(p \to q), \pm q\}$. Here, the premise issues might be $\{\pm p\}$ and $\{\pm(p \to q)\}$, and the conclusion issue might be $\{\pm q\}$. The intuition is that the former might somehow be more fundamental than the latter, so that an agent's judgments on the latter may be derived from the agent's judgments on the former.

To define a *premise-based aggregation rule*, we require two preliminary definitions. For each premise issue $Z \in \mathcal{Z}_{\text{prem}}$, we introduce a *local aggregation rule ("premise aggregator") F_Z* which assigns to each combination of individual judgments on Z a collective judgment on Z. Formally, F_Z is a function from $\mathcal{J}_Z^n$ to $\mathcal{J}_Z$, where $\mathcal{J}_Z$ is the set of all *locally complete and consistent* judgments on Z, i.e., $\mathcal{J}_Z = \{\{p\}, \{\neg p\}\}$ for the binary issue $Z = \{\pm p\}$, assuming p is contingent. In the classical premise-based aggregation rule, each F_Z is majority rule, if n is odd.

To derive the judgments on all conclusion issues, we employ a *consequence rule*, defined as a function Cn that assigns to each set of (already accepted) propositions $J \subseteq X$ another set $Cn(J) \subseteq X$ of propositions that are the "consequences" of J. In the classical case, $Cn(J)$ simply consists of all propositions p in X that are logically entailed by J in the sense that the negation of p is inconsistent with J.

For any profile of individual judgment sets, we now arrive at the overall collective judgment set by

- first aggregating the individual judgments on all the premises, using the given local aggregation rules, and
- then deriving their consequences for all other propositions, using the given consequence rule.

Formally, we define our premise-based aggregation rule on the domain of all profiles of judgment sets $J \subseteq X$ that are *classically rational on the premises*, i.e., $J \cap Z \in \mathcal{J}_Z$ for all $Z \in \mathcal{Z}_{\text{prem}}$. Let $\hat{\mathcal{J}}$ be the set of all such judgment sets. (This is a superset of $\mathcal{J}$.) For any profile $\langle J_1, ..., J_n \rangle \in \hat{\mathcal{J}}^n$, we let

$$F(J_1, ..., J_n) = \bigcup_{Z \in \mathcal{Z}} J_Z,$$

where, for each binary issue $Z \in \mathcal{Z}$,

$$J_Z = \begin{cases} F_Z(J_1 \cap Z, ..., J_n \cap Z) & \text{if } Z \in \mathcal{Z}_{\text{prem}}, \\ Cn(\bigcup_{Z' \in \mathcal{Z}_{\text{prem}}} J_{Z'}) \cap Z & \text{if } Z \in \mathcal{Z}_{\text{conc}}. \end{cases}$$

To illustrate this definition, consider the agenda $X = \{\pm p, \pm(p \to q), \pm q\}$ with $\{\pm p\}$ and $\{\pm(p \to q)\}$ designated as the premise issues, and suppose the individual judgments are as shown in Table 2. If the premise-based rule is the classical one, where each premise aggregator F_Z is the majority rule and the consequence rule Cn is the classical one, the collective judgment set will be $\{p, p \to q, q\}$. Propositions p and $p \to q$ will each be accepted by aggregating the individual judgments on those propositions, and proposition q will be accepted by logical inference. It is evident that this aggregation rule violates systematicity, by treating premises and conclusions differently and also by determining the collective judgments on all conclusions in a non-propositionwise-independent way.

Table 2: A premise-based rule illustrated

	p	$p \to q$	q
Individual 1	Yes	Yes	Yes
Individual 2	Yes	No	No
Individual 3	No	Yes	No
Premise-based rule	Yes	Yes	Yes

Next we introduce the idea of premise-based revision. Here, we also need some preliminary definitions. For each premise issue $Z \in \mathcal{Z}_{\text{prem}}$, we introduce a *local revision* *operator ("premise revisor")*, denoted $|_Z$, just for that issue. Formally, this is a function (from $2^Z \times X$ into 2^Z) which assigns to any pair (L, p) of an initial local judgment L on issue Z (formally $L \subseteq Z$) and a learnt proposition $p \in X$ a new local judgment on issue Z, denoted $L|_Z p$. As issue Z is of the form $\{\pm p\}$, any local judgment on Z must be of the form $\emptyset, \{p\}, \{\neg p\}, \{p, \neg p\}$. Of these, the first would correspond to withholding judgment on Z, the last would be inconsistent, and only the middle two would encode a locally complete and consistent judgment on issue Z (assuming neither p nor $\neg p$ is contradictory). To derive the revised judgments on all conclusion issues, we employ again our consequence rule Cn, which allows us to assign to each set of (already revised) propositions $J \subseteq X$ the set of propositions that are its consequences, $Cn(J) \subseteq X$.

For any initial judgment set and any newly learnt proposition, the premise-based revision operator now arrives at the revised judgment set by

- first revising the judgments on all the premises, using the given local revision operators, and
- then deriving their consequences for all other propositions, using the given consequence rule.

Formally, for any initial judgment set $J \subseteq X$ and any learnt proposition $p \in X$, the revised judgment set $J|p$ is the union

$$J|p = \bigcup_{Z \in \mathcal{Z}} J_Z$$

of revised local judgment sets $J_Z \subseteq Z$ for all binary issues $Z \in \mathcal{Z}$, where

$$J_Z = \begin{cases} (J \cap Z)|_Z p & \text{if } Z \in \mathcal{Z}_{\text{prem}}, \\ Cn(\bigcup_{Z' \in \mathcal{Z}_{\text{prem}}} J_{Z'}) \cap Z & \text{if } Z \in \mathcal{Z}_{\text{conc}}. \end{cases}$$

The premise-based revision operator is often neither regular nor rationality-preserving, as defined earlier, but any plausible premise-based revision operator satisfies weaker versions of these conditions. In particular, it satisfies *regularity on premises*, in the sense that it satisfies our two regularity conditions, successfulness and conservativeness, restricted to the premises. *Successfulness on premises* means that $p \in J|p$ whenever $p \in X_{\text{prem}}$, and *conservativeness on premises* means that if $p \in J \cap X_{\text{prem}}$, then J and $J|p$ coincide on the premises, i.e., $J \cap X_{\text{prem}} = (J|p) \cap X_{\text{prem}}$. The first of these properties permits that one does not incorporate a newly learnt conclusion proposition in one's revised judgments; rather, one always builds up one's revised judgments from one's judgments on the premises. And the second property permits that if one learns – or is reminded of – an already known premise, one might still change one's judgments on some conclusion, for instance by recognizing certain hitherto unacknowledged consequences of one's existing premise judgments. The premise-based revision operator may fail to be rationality-preserving insofar as a complete and consistent pre-revision judgment set does not always guarantee a complete and consistent post-revision judgment set. Whether or not it does depends very much on the nature of the subagenda of premises and the nature of the consequence rule. If consequence is defined classically, for instance, then the completeness of the revised judgments depends on whether complete judgments on the premises always logically settle all conclusion propositions; and if the premise issues are logically dependent, then the consistency of the revised judgments may be threatened by the fact that premise-based revision operates independently on each premise issue.

We are now in a position to state our possibility result. Call a revision operator *idempotent* if $(J|p)|p = J|p$ for all $J \subseteq X$ and all $p \in X$ ("learning the same information again does not change one's judgments"). Idempotence is much less demanding than full-blown regularity.

Theorem 3. If the revision operator is premise-based and idempotent, then all premise-based aggregation rules with unanimity-preserving premise aggregators (and with the same premises and consequence rule as in revision) are dynamically rational.

Here, a premise aggregator F_Z is *unanimity-preserving* if $F(L, ..., L) = L$ for any unanimous local judgment profile $(L, ..., L)$ on the premise issue Z (i.e., $L \subseteq Z$).

In fact, we can go beyond Theorem 3 and show that, in important special cases, premise-based rules are the *only* dynamically rational aggregation rules with respect to a premise-based revision operator. To state this uniqueness result, we need to introduce two other conditions on the aggregation rule, which replace our original monotonicity and systematicity conditions, neither of which is generally satisfied by a premise-based rule. The first condition is a global version of monotonicity which replaces the focus on accepted propositions with a focus on accepted judgment sets:

Global monotonicity: Additional individual support for a "winning" judgment set does not overturn the outcome, i.e., if any profile $\langle J_1, ..., J_n\rangle \in \mathcal{D}$ is modified into another profile $\langle J_1, ..., J, ..., J_n\rangle \in \mathcal{D}$ by replacing one of the J_is with $J = F(J_1, ..., J_n)$, then $F(J_1, ..., J, ..., J_n) = J$.

To state the second condition, note that any given subagenda of premises X_{prem} induces a *relevance relation* between propositions: premises are relevant to conclusions, but not vice versa. More precisely, the only proposition relevant to any premise $p \in X_{\text{prem}}$ is p itself, while the propositions relevant to any conclusion $p \in X_{\text{conc}}$ are all premises. Formally, if $\mathcal{R}(p)$ denotes the set of propositions relevant to p, we have

$$\mathcal{R}(p) = \begin{cases} \{p\} & \text{if } p \in X_{\text{prem}}, \\ X_{\text{prem}} & \text{if } p \in X_{\text{conc}}. \end{cases}$$

Now our condition that replaces systematicity is the following (Dietrich 2015):

Independence of irrelevant propositions: The collective judgment on each proposition depends only on individual judgments on relevant propositions. Formally, for any proposition $p \in X$ and any profiles $\langle J_1, ..., J_n \rangle, \langle J'_1, ..., J'_n \rangle \in \mathcal{D}$, if, for all $i \in N$, $J_i \cap \mathcal{R}(p) = J'_i \cap \mathcal{R}(p)$, then $p \in J \Leftrightarrow p \in J'$, where $J = F(J_1, ..., J_n)$ and $J' = F(J'_1, ..., J'_n)$.

Global monotonicity and independence of irrelevant propositions jointly weaken the conjunction of monotonicity and systematicity used in our impossibility theorem. Here is the uniqueness theorem:

Theorem 4. If the revision operator is premise-based, idempotent, and regular on premises, then the premise-based aggregation rules with unanimity-preserving premise aggregators (and with the same premises and consequence rule as in revision) are the only dynamically rational aggregation rules F from $\hat{\mathcal{J}}^n$ into $\hat{\mathcal{J}}$ satisfying independence of irrelevant propositions and global monotonicity.

Insofar as premise-based judgment aggregation has been prominently discussed in the literature, the present possibility and uniqueness results should be interesting. Indeed, Theorem 4's conditions on the aggregation rule seem eminently reasonable. In particular, independence of irrelevant propositions is arguably much more plausible than systematicity, and global monotonicity is a very plausible condition too. The conditions on the revision operator – idempotence and regularity on premises – are reasonable as well. However, the cost of the present possibility, as we have already noted, is that the revision operator is not generally fully regular or rationality-preserving. We leave it an open question for further discussion whether this cost is worth bearing.

To conclude, the lesson of this paper is that it is surprisingly difficult to achieve rationality at the collective level merely through the aggregation of individual attitudes. While this point is well known in the case of static rationality, our results are the first in the present judgment-aggregation framework to extend the point to dynamic rationality.

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A Proof of both impossibility theorems

A.1 Theorem 1

We first prove Theorem 1. One could easily prove Theorem 1 as a corollary of Theorem 2, but we here provide a direct, self-contained proof. The proof begins with a lemma. Recall that an aggregation rule F *preserves unanimity* if $F(J, ..., J) = J$ for all unanimous profiles $\langle J, ..., J\rangle$ in its domain. A judgment set is *weakly consistent* if it contains no pair $p, \neg p \in X$ (i.e., is not 'drastically inconsistent'). An aggregation rule F *guarantees* some condition on judgment sets (e.g., weak consistency) if $F(J_1, ..., J_n)$ satisfies the condition for each profile $\langle J_1, ..., J_n\rangle$ in the domain.

Lemma 1 *If a unanimity-preserving systematic aggregation rule with universal domain (e.g., a uniform quota rule) is dynamically rational with respect to a regular rationality-preserving revision operator, then it guarantees weak consistency.*

Proof. Let F be as specified. We may assume without loss of generality that X contains a contingent proposition; otherwise there would exist only one (unanimous) profile in $\mathcal{J}^n$, and weak consistency would follow from unanimity preservation.

Consider a profile $\langle J_1, ..., J_n\rangle \in \mathcal{J}^n$ and a $p \in F(J_1, ..., J_n)$. We show that $\neg p \notin F(J_1, ..., J_n)$. By assumption, there is a contingent $q \in X$. As $\neg q$ is non-contradictory, some $J \in \mathcal{J}$ contains $\neg q$. Meanwhile $q \in J|q$ by successfulness, and $J|q \in \mathcal{J}$ by rationality-preservation and q's consistency. Construct the profile $\langle J'_1, ..., J'_n\rangle \in \mathcal{J}^n$ in which

$$J'_i = \begin{cases} J|q & \text{if } p \in J_i \\ J & \text{if } p \notin J_i. \end{cases}$$

Note that, for all individuals i, $p \in J_i \Leftrightarrow q \in J'_i$, or equivalently, $\neg p \in J_i \Leftrightarrow \neg q \in J'_i$. So, by systematicity, it suffices to show that $\neg q \notin F(J'_1, ..., J'_n)$. By rationality-preservation, $\langle J'_1|q, ..., J'_n|q\rangle \in \mathcal{J}^n$. So, by dynamic rationality, $F(J'_1|q, ..., J'_n|q) = F(J'_1, ..., J'_n)|q$. In this equation, the left side equals $F(J|q, ..., J|q)$ (because $(J|q)|q = J|q$ by regularity), which in turn equals $J|p$ by unanimity preservation; and the right side equals $F(J'_1, ..., J'_n)$, by conservativeness and the fact that $q \in F(J'_1, ..., J'_n)$. So, $J|q = F(J'_1, ..., J'_n)$. Hence, $\neg q \notin F(J'_1, ..., J'_n)$. ■

Proof of Theorem 1. Let X be non-simple; so we may pick a minimal inconsistent set $Y \subseteq X$ with $|Y| \geq 3$. Let F be a uniform quota rule on $\mathcal{J}^n$ with some acceptance threshold $m < n$. Fix a regular rationality-preserving revision operator. For a contradiction, assume F is dynamically rational. Then, by Lemma 1, F guarantees weak consistency; so $m > \frac{n}{2}$.

For each $y \in Y$, fix a rational judgment set $J_{\neg y} \in \mathcal{J}$ such that $Y \backslash \{y\} \subseteq J_{\neg y}$. Pick a $p \in Y$. Since $J_{\neg p}|p$ cannot contain all $y \in Y$ (as revision preserves rationality) but contains p (as revision is successful), there is some $q \in Y \backslash \{p\}$ such that $q \notin J_{\neg p}|p$. As $|Y| \geq 3$, we may pick a third proposition $r \in Y \backslash \{p, q\}$.

Let $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ be a profile in which some $n - m$ individuals i hold $J_i = J_{\neg p}$, other $n - m$ individuals i hold $J_i = J_{\neg q}$, and all remaining individuals i hold $J_i = J_{\neg r}$. As p and q are each accepted by m individuals, $p, q \in F(J_1, ..., J_n)$. Consider the revised profile $\langle J_1|p, ..., J_n|p \rangle$. As $q \notin J_{\neg p}|p$, and as by regularity $J_{\neg q}|p = J_{\neg q}$ and $J_{\neg r}|p = J_{\neg r}$, in the new profile only the individuals who used to hold $J_{\neg r}$ accept q; so q is accepted by $n - 2(n - m) = 2m - n < m$ individuals. So $q \notin F(J_1|p, ..., J_n|p)$. Now, $F(J_1, ..., J_n)$ equals $F(J_1, ..., J_n)|p$ because it contains p and revision is regular, and differs from $F(J_1|p, ..., J_n|p)$ because it contains q while $F(J_1|p, ..., J_n|p)$ does not. Therefore $F(J_1|p, ..., J_n|p) \neq F(J_1, ..., J_n)|p$. ■

A.2 Theorem 2

We now prove Theorem 2, in a slightly stronger version that weakens the (already weak) condition of non-imposition to non-absurdity. Non-imposition forbids that the collective judgment set is always the same. Non-absurdity merely forbids that the collective judgment set is always the entire agenda X (an absurd judgment set).

The proof of Theorem 2 uses again Lemma 1, but it also uses the following additional lemma.

Lemma 2 *The aggregation conditions in Theorem 2 with non-imposition weakened to non-absurdity (and with dynamic rationality defined with respect to a regular rationality-preserving revision operator) imply unanimity-preservation.*

Proof. Let F be an aggregation rule satisfying these conditions, with regular rationality-preserving revision. We show unanimity-preservation. By systematicity, it suffices to show that N is a winning coalition and $\varnothing$ is a losing coalition.

Claim 1: The full coalition N is winning.

Consider any rational profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ and any p contained in all J_i. We must show that $F(J_1, \ldots, J_n)$ contains p. As revision is conservative, $\langle J_1|p, ..., J_n|p \rangle = \langle J_1, ..., J_n \rangle$. So, by dynamic rationality, $F(J_1, \ldots, J_n)|p = F(J_1, \ldots, J_n)$. The left side contains p by successfulness of revision. So $F(J_1, \ldots, J_n)$ contains p. Q.e.d.

Claim 2: There is a non-tautological $p \in X$ and a profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ such that $p \notin F(J_1, ..., J_n)$.

By non-absurdity, there is a $p \in X$ and a profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ such that $p \notin F(J_1, ..., J_n)$. Assume for a contradiction that p is tautological. Then p belongs to all J_i. So $\langle J_1, ..., J_n \rangle = \langle J_1|p, ..., J_n|p \rangle$, as revision is conservative. Hence, by dynamic rationality $F(J_1, \ldots, J_n)|p = F(J_1, \ldots, J_n)$. Noting that $p \in F(J_1, \ldots, J_n)|p$ (as revision is successful), it follows that $p \in F(J_1, ..., J_n)$, a contradiction. Q.e.d.

Claim 3: The empty coalition $\varnothing$ is not winning.

Pick p and $\langle J_1, ..., J_n \rangle$ as in Claim 2. As p is non-tautological, there is a rational profile $J \in \mathcal{J}$ such that $p \notin J$. Since $p \notin F(J_1, ..., J_n)$, and since each individual accepting p in the profile $\langle J, ..., J \rangle \in \mathcal{J}^n$ (namely, no-one) accepts p in $\langle J_1, ..., J_n \rangle$, we have $p \notin F(J, ..., J)$, by monotonicity. So the empty coalition $\varnothing$ $(= \{i : p \in J_i\})$ is not winning. ■

Proof of Theorem 2. Let X be non-simple. For a contradiction, assume F is an aggregation rule satisfying all mentioned conditions, with dynamic consistency defined with respect to a given regular rationality-preserving revision operator. By Lemmas 1 and 2, F is unanimity-preserving and guarantees weak consistency.

By non-simplicity, we may pick a minimal inconsistent set $Y \subseteq X$ with $|Y| \geq 3$. For each $y \in Y$, fix a $J_{\neg y} \in \mathcal{J}$ such that $Y \backslash \{y\} \subseteq J_{\neg y}$. Pick a $p \in Y$. Since $J_{\neg p}|p$ cannot contain all $y \in Y$ (as revision preserves rationality) but contains p (as revision is successful), there is some $q \in Y \backslash \{p\}$ such that $q \notin J_{\neg p}|p$. As $|Y| \geq 3$, we may pick a third proposition $r \in Y \backslash \{p, q\}$.

By systematicity, F is given by its winning coalitions. Note the following:

- N is winning while $\varnothing$ is not winning, by unanimity-preservation.
- Supersets of winning coalitions are winning, i.e., whenever $C \subseteq N$ is winning, so is any $C' \subseteq N$ such that $C \subseteq C'$. This follows from monotonicity.
- Any two winning coalitions C, C' have non-empty intersection. Otherwise $N \backslash C \supseteq C'$, so that $N \backslash C$ would be winning by monotonicity; but then we would have two complementary winning coalitions (C and $N \backslash C$), which would contradict weak consistency.
- There exist at least two *minimal* winning coalitions. Otherwise the set of winning coalitions would be a filter over N, implying oligarchy.

Pick two distinct minimal winning coalition C and C'. Construct a profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ by letting

$$J_i = \begin{cases} J_{\neg p} & \text{if } i \in N \backslash C \\ J_{\neg q} & \text{if } i \in C \backslash C' \\ J_{\neg r} & \text{if } i \in C \cap C'. \end{cases}$$

As p and q are accepted by winning coalitions (namely by C and by $N \backslash (C \backslash C') \supseteq C'$, respectively), $p, q \in F(J_1, ..., J_n)$. Consider the revised profile $\langle J_1|p, ..., J_n|p\rangle$. As $q \notin J_{\neg p}|p$ and as (by conservativeness of revision) $J_{\neg q}|p = J_{\neg q}$ and $J_{\neg r}|p = J_{\neg r}$, in the new profile q is accepted only by those individuals who used to submit $J_{\neg r}$, hence by the coalition $C \cap C'$. This coalition is not winning because (as $C \cap C' \neq \varnothing$) it is a *strict* subset of a *minimal* winning coalition (i.e., of C or C'). So $q \notin F(J_1|p, ..., J_n|p)$. Now, $F(J_1, ..., J_n)$ equals $F(J_1, ..., J_n)|p$ (as it contains p and as revision is conservative) and it differs from $F(J_1|p, ..., J_n|p)$ because it contains q while $F(J_1|p, ..., J_n|p)$ does not. Therefore, $F(J_1|p, ..., J_n|p) \neq F(J_1, ..., J_n)|p$. ■

B Proof of the possibility claims in Section 6

B.1 Possibilities without universal domain

In the main text, we have discussed two types of aggregation rules satisfying all conditions in Theorem 2 except universal domain. The second type requires formal elaboration. There, we consider a fixed linear order $\leq$ of the propositions, representing for instance a political left-to-right order or the propositions. Recall that $\mathcal{J}_{\leq}$ denotes the set of those rational judgment sets $J \in \mathcal{J}$ which are *single-plateaued* with respect to $\leq$, as defined earlier. Recall also that the order $\leq$ induces a natural revision rule, as defined above.[1] This revision operator is obviously regular and rationality-preserving. As long as n is odd, majority rule restricted to $\mathcal{J}_{\leq}^n$ satisfies all aggregation conditions of Theorem 2 except universal domain. This is obvious for most aggregation conditions, but requires a proof for dynamic rationality.

Proposition 2 *If n is odd, majority rule on the restricted domain $\mathcal{J}_{\leq}^n$ is dynamically rational with respect to the above revision operator.*

Proof. Assume n is odd, F is majority rule on $\mathcal{J}_{\leq}^n$, and revision is defined as above. To prove dynamic rationality, consider any $\langle J_1, ..., J_n\rangle \in \mathcal{J}_{\leq}^n$ and $p \in X$ such that $\langle J_1|p, ..., J_n|p\rangle \in \mathcal{J}_{\leq}^n$. For all $J \in \mathcal{J}$, write $\min J$ for J's minimal element with respect to $\leq$. For simplicity, assume that $i < j \Rightarrow \min J_i \leq \min J_j$. Assuming this condition is no loss of generality, because the condition can always be enforced by reordering the profile appropriately (reordering makes no difference since majority rule is anonymous).

[1] Our above definition of $J|p$ could be generalised slightly, as follows. If $J \in \mathcal{J}_{\leq}$ and some $J' \in \mathcal{J}_{\leq}$ contains p (first bullet point), the definition of $J|p$ remains unchanged. Otherwise (second bullet point), $J|p$ can be defined arbitrarily, only subject to respecting the regularity conditions that $p \in J|p$ and that $J|p = J$ if $p \in J$, and the rationality-preservation condition that $J|p \in \mathcal{J}$ if $J \in \mathcal{J}$ and p is non-contradictory. Our proof holds for this general definition.

As the profile is single-plateaued and rational, it is unidimensionally aligned (see Dietrich and List 2010). Unidimensional alignment means that there exists a permutation $(i_1, ..., i_n)$ of the individuals such that each proposition $q \in X$ is accepted either by a 'left-segment' of individuals (i.e., $\{i : q \in J_i\} = \{i_1, ..., i_k\}$ for some $k \in \{0, ..., n\}$) or by a 'right-segment' of individuals (i.e., $\{i : q \in J_i\} = \{i_k, ..., i_n\}$ for some $k \in \{1, ..., n+1\}$); a consequence is that the majority judgment set is the judgment set of the median individual, i.e.,

$$F(J_1, ..., J_n) = J_{i_{(n+1)/2}}$$

(List 2003). A permutation $(i_1, ..., i_n)$ with the mentioned property is called a *structuring order*, and the profile $\langle J_1, ..., J_n \rangle$ is more explicitly called unidimensionally aligned 'with respect to $(i_1, ..., i_n)$'. Our initial assumption on the order of the judgment sets in $\langle J_1, ..., J_n \rangle$ yields a natural structuring order:

Claim 1: $\langle J_1, ..., J_n \rangle$ is unidimensionally aligned with respect to the structuring order $(1, ..., n)$. In particular,

$$F(J_1, ..., J_n) = J_{(n+1)/2}. \tag{1}$$

Write $X = \{p_1, ..., p_{|X|}\}$ where $p_1 < p_2 < \cdots < p_{|X|}$. Consider any $q \in X$. There are two cases.

- *Case 1:* $q \in \{p_1, ..., p_{|X|/2}\}$, i.e., q is 'more to the left'. We show that $\{i : q \in J_i\} = \{1, ..., k\}$ for some $k \in \{0, ..., n\}$. To prove this, we consider an individual i such that $q \in J_i$, and show for any given other individual $j < i$ that again $q \in J_j$. This follows from three facts. First, $\min J_j \leq q$, because $\min J_j \leq \min J_i$ (as $j < i$) and $\min J_i \leq p$ (as $p \in J_i$). Second, $q \leq \max J_j$, because J_j contains $\frac{|X|}{2}$ propositions (by rationality) while there are less than $\frac{|X|}{2}$ propositions to the left of q (as $q \in \{p_1, ..., p_{|X|/2}\}$). Third, J_j is an 'interval' or 'plateau', i.e., contains all propositions between $\min J_j$ and $\max J_j$, by single-plateauedness.
- *Case 2:* $q \in \{p_{|X|/2+1}, ..., p_{|X|}\}$, i.e., q is 'more to the right'). We show that $\{i : q \in J_i\} = \{k, ..., n\}$ for some $k \in \{1, ..., n+1\}$. To this end, we consider an individual i such that $q \in J_i$, and show for any other individual $j > i$ that again $q \in J_j$. This holds because, for reasons analogous to those in Case 1, $q \leq \max J_j$, $\min J_j \leq q$, and J_j is an 'interval'. Q.e.d.

Claim 2: The set $\mathcal{J}_{\leq,p} := \{J \in \mathcal{J}_{\leq} : p \in J\}$ is non-empty.

Recall that $\langle J_1|p, ..., J_n|p \rangle \in \mathcal{J}_{\leq}^n$ and each $J_i|p$ contains p. So, $\langle J_1|p, ..., J_n|p \rangle \in \mathcal{J}_{\leq,p}^n$, whence $\mathcal{J}_{\leq,p} \neq \varnothing$. Q.e.d.

Claim 3: The revised judgment profile is

$$(J_1|p, ..., J_n|p) = \begin{cases} (J_1, ..., J_k, J, ..., J) & \text{if } p \in \{p_1, ..., p_{|X|/2}\} \\ (J', ..., J', J_{k'}, ..., J_n) & \text{if } p \in \{p_{|X|/2+1}, ..., p_{|X|}\} \end{cases}$$

where

- $k = \max\{i : p \in J_i\}$, interpreted as 0 if $\{i : p \in J_i\} = \varnothing$,
- J is the right-most judgment set in $\mathcal{J}_{\leq,p}$ ($\neq \varnothing$), i.e., $J \in \mathcal{J}_{\leq,p}$ and $\min \tilde{J} \leq \min J$ for all $\tilde{J} \in \mathcal{J}_{\leq,p}$,
- $k' = \min\{i : p \in J_i\}$, interpreted as $n+1$ if $\{i : p \in J_i\} = \varnothing$,
- J' is the left-most judgment set in $\mathcal{J}_{\leq,p}$ ($\neq \varnothing$), i.e., $J' \in \mathcal{J}_{\leq,p}$ and $\min J' \leq \min \tilde{J}$ for all $\tilde{J} \in \mathcal{J}_{\leq,p}$.

This claim follows from the definition of the revision operator and the fact that in the profile $\langle J_1, ..., J_n \rangle$ the judgment sets are ordered such that $i < j \Rightarrow \min J_i \leq \min J_j$. For instance, assume $p \in \{p_1, ..., p_{|X|/2}\}$. Then no judgment set J_i is 'to the left' of p (as $|J_i| = |X|/2$). Those J_i which already contain p are unchanged: $J_i|p = J_i$. Those J_i which do not contain p lie 'to the right' of p, so that their revision 'shifts' them minimally to the left such that p is accepted: $J_i|p = J$. Q.e.d.

Claim 4: $\langle J_1|p, ..., J_n|p \rangle$ is again unidimensionally aligned with respect to the structuring $(1, ..., n)$. In particular,

$$F(J_1|p, ..., J_n|p) = J_{(n+1)/2}|p. \tag{2}$$

Claim 3 implies that the judgment sets in the revised profile $\langle J_1|p, ..., J_n|p \rangle$ have the analogous property to that of in the original profile: $i < j \Rightarrow \min(J_i|p) \leq \min(J_j|p)$. So, by an argument analogous to that used to prove Claim 1, $\langle J_1|p, ..., J_n|p \rangle$ is a unidimensionally aligned profile with structuring order $(1, ..., n)$. Q.e.d.

By (1) in Claim 1 and (2) in Claim 4, $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$. ■

B.2 Possibilities without non-imposition

As noted, there is a single aggregation rule satisfying all conditions of Theorem 2 except non-imposition: the *absurd* rule, which maps each $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ to the judgment set $F(J_1, ..., J_n) = X$. The absurd rule obviously satisfies all other conditions. In particular, dynamic rationality is trivially satisfied with respect to any conservative revision operator, because conservativeness prevents the collective from ever revising its degenerate judgment set $J = X$.

But why do other constant aggregation rules on $\mathcal{J}^n$ fail to satisfy all other conditions? While this fact already follows from our proof in Appendix A.2, let us now give an intuition. Consider a constant rule on $\mathcal{J}^n$ which always generates some given judgment set $J \neq X$. Pick a $p \in X \backslash J$, and let p be contingent for the sake of this illustration. Since p is collectively rejected *regardless* of which individuals accept p, no coalition whatsoever is winning for p. Therefore, supposing systematicity, no coalition is winning for any proposition in X. So no proposition is ever collectively accepted: $J = \varnothing$. But then dynamic rationality fails, because whenever the individuals learn some (contingent) proposition p, then the revised judgment profile still aggregates into $J = \varnothing$, although dynamic rationality would have required the collective to come to acquire the judgment set $\varnothing|p$, which contains p, assuming revision is successful.

B.3 Possibilities without monotonicity

In identifying a non-monotonic escape route, we have limited attention to agendas with two structural properties, and introduced a particular revision operator for such agendas. We have claimed that these revision operators obey our requirements, and that so-called *parity rules* satisfy all conditions of Theorem 2 except monotonicity. Both claims are now established formally.

Proposition 3 *For agendas with both properties, the revision operator defined earlier is regular and rationality-preserving.*

Proof. Let X satisfy both conditions. The relevant revision operator is obviously regular. To show that it preserves rationality, assume J is rational and p is non-contradictory. We must show that $J|p$ is rational. This is obvious if $p \in J$, as then $J|p = J$. Henceforth let $p \notin J$. So $J|p = (X_p \backslash J) \cup (J \backslash X_p)$, where '$X_p$' is the earlier-defined subagenda. Since J contains exactly one member of each pair $q, \neg q \in X$, so does $J|p$. It thus remains to show that $J|p$ is consistent. For a contradiction, let $J|p$ be inconsistent. Pick a minimal inconsistent subset Y of $J|p$. Noting that p is contingent (it is non-contradictory by assumption and non-tautological by $p \notin J$), it follows that $|Y \cap X_p| \in \{0, 2, 4, ...\}$. So, since X is affine (i.e., not pair-negatable), the set Y' arising from Y by negating the members of $Y \cap X_p$ is again inconsistent. So, as $Y' \subseteq J$, also J is inconsistent, a contradiction. ■

Proposition 4 *If the agenda X has both properties (and contains at least one contingent proposition, e.g., is non-simple), then all parity rules with $|M| \neq 1$ satisfy each condition of the theorem except monotonicity.*

Proof. Let X be as specified. Consider the parity rule F whose (odd-sized) subgroup $M \subseteq N$ satisfies $|M| \neq 1$. Clearly, F is universal, non-oligarchic, non-constant, systematic, and non-monotonic, where non-oligarchy and non-monotonicity hold because $|M| \neq 1$ (and because X contains a contingent proposition). To prove dynamic rationality, consider a profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ and a $p \in X$ such that $\langle J_1|p, ..., J_n|p \rangle \in \mathcal{J}^n$ (p

is non-contradictory because $J_1|p$ is rational and contains p). We fix a $q \in X$ and must show that

$$q \in F(J_1|p, ..., J_n|p) \Leftrightarrow q \in F(J_1, ..., J_n)|p. \tag{3}$$

Note that $F(J_1, ..., J_n) \in \mathcal{J}$ since parity rules guarantee rationality for affine, i.e., non-pair-negatable, agendas (Dokow and Holzman 2010).

Case 1: $q \in X \backslash X_p$. Then, as $J_1, ..., J_n$ are rational, by definition of revision we have

$$q \in J_i \Leftrightarrow q \in J_i|p \text{ for } i = 1, ..., n,$$

which by independence of parity rules implies

$$q \in F(J_1, ..., J_n) \Leftrightarrow q \in F(J_1|p, ..., J_n|p).$$

Analogously, as $F(J_1, ..., J_n)$ is rational, by definition of revision we have

$$q \in F(J_1, ..., J_n) \Leftrightarrow q \in F(J_1, ..., J_n)|p.$$

These two equivalences together imply (3).

Case 2: $q \in X_p$. By definition of revision, for all $i \in M$, as $J_i \in \mathcal{J}$,

$$q \in J_i|p \Leftrightarrow \begin{cases} q \in J_i & \text{if } p \in J_i \\ q \notin J_i & \text{if } p \notin J_i, \end{cases} \tag{4}$$

and analogously, as $F(J_1, ..., J_n) \in \mathcal{J}$,

$$q \in F(J_1, ..., J_n)|p \Leftrightarrow \begin{cases} q \in F(J_1, ..., J_n) & \text{if } p \in F(J_1, ..., J_n) \\ q \notin F(J_1, ..., J_n) & \text{if } p \notin F(J_1, ..., J_n). \end{cases} \tag{5}$$

It will prove useful to prove a simple combinatorial fact:

$$\text{finite sets } S \text{ and } S' \text{ have same parity if and only if } \left|S \triangle S'\right| \text{ is even.} \tag{6}$$

Here, the parity of a set is 'even' or 'odd', depending on whether its cardinality is even or odd, and $S \triangle S'$ denotes the symmetric difference $(S \backslash S') \cup (S' \backslash S)$. The equivalence (6) holds because, for any finite sets S and S', firstly S and S' have same parity if and only if $|S| + |S'|$ is even, and secondly $|S| + |S'|$ is even if and only if $|S \triangle S'|$ is even as

$$|S| + \left|S'\right| = \left|S \cap S'\right| + \left|S \backslash S'\right| + \left|S \cap S'\right| + \left|S' \backslash S\right| = 2\left|S \cap S'\right| + \left|S \triangle S'\right|.$$

For all $r \in X$, let $M_r := \{i \in M : r \in J_i\}$ and $M'_r = \{i \in M : r \in J_i|p\}$. By (4),

$$M_q \triangle M'_q = M \backslash M_p. \tag{7}$$

We consider two subcases.

Subcase 2.1: $|M_p|$ is odd. Then, by definition of parity rules, $p \in F(J_1, ..., J_n)$, so that by (5)

$$q \in F(J_1, ..., J_n)|p \Leftrightarrow q \in F(J_1, ..., J_n). \tag{8}$$

As $|M|$ and $|M_p|$ are odd, $|M \backslash M_p|$ $(= |M| - |M_p|)$ is even. Hence, by (7), $\left|M_q \bigtriangleup M'_q\right|$ is even, so that by (6) M_q and M'_q have same parity. Thus, by definition of parity rules,

$$q \in F(J_1, ..., J_n) \Leftrightarrow q \in F(J_1|p, ..., J_n|p).$$

This equivalence and the equivalence (8) imply (3).

Subcase 2.2: $|M_p|$ is even. Then, firstly, $p \notin F(J_1, ..., J_n)$, so that by (5)

$$q \in F(J_1, ..., J_n)|p \Leftrightarrow q \notin F(J_1, ..., J_n). \tag{9}$$

As $|M|$ is odd and $|M_p|$ is even, $|M \backslash M_p|$ $(= |M| - |M_p|)$ is odd. Hence, by (7), $\left|M_q \bigtriangleup M'_q\right|$ is odd, and so by (6) M_q and M'_q have opposed parity. Therefore, by definition of parity rules,

$$q \notin F(J_1, ..., J_n) \Leftrightarrow q \in F(J_1|p, ..., J_n|p).$$

Combining this equivalence with (9), we again obtain (3). ■

B.4 Possibilities without non-oligarchy

We have specified two types of aggregation rule that satisfy all conditions of Theorem 2 except non-oligarchy. The first of these oligarchic escape routes are trivial: they are the dictatorships. We here focus on the second, less trivial, oligarchic possibility. This possibility was restricted to a special agenda (of the form $X = \{\pm p_1, \pm p_2, \pm p_3\}$ with certain logical interconnections) and a special revision operator, as defined above. We now formally establish that this revision operator indeed has the desirable properties, and that oligarchies become dynamically rational (they obviously satisfy the other conditions in Theorem 2 except non-oligarchy).

The sets $J_p \in \mathcal{J}$ (for $p \in X$ and $J \in \mathcal{J}$) are defined as before, and $\mathcal{J}^+ \subseteq \mathcal{J}$ still denotes the set of consistent and deductively closed judgment sets. Recall that for all $J \in \mathcal{J}^+$

$$J|p = \cap_{J' \in \mathcal{J}: J \subseteq J'} J'_p, \tag{10}$$

which in the special case of a rational $J \in \mathcal{J}$ implies

$$J|p = J_p \text{ if } J \in \mathcal{J}. \tag{11}$$

Proposition 5 *The specified revision operator for the special agenda X is regular and rationality-preserving.*

Proof. Revision is rationality-preserving by definition. To show that revision is successful, consider any $p \in X$ and $J \subseteq X$. We show that $p \in J|p$. If $J \notin \mathcal{J}^+$, then $p \in J|p$ by assumption. If $J \in \mathcal{J}^+$, then $p \in J|p$ because in (10) each J'_p contains p.

To finally show that revision is conservative, consider any $p \in J \subseteq X$. We prove that $J|p = J$. If $p \notin \mathcal{J}^+$, then this again holds by assumption. If $J \in \mathcal{J}^+$, then it holds because

$$J|p = \cap_{J' \in \mathcal{J} : J \subseteq J'} J'_p = \cap_{J' \in \mathcal{J} : J \subseteq J'} J' = J,$$

where the first equality holds by definition of $J|p$, the second because each J'_p equals J' (as $p \in J'$), and the third by Lemma 3 below. ■

The following is a general logical fact about deductively closed judgment sets, which does not depend on our specific agenda.

Lemma 3 *For an arbitrary agenda X, the consistent and deductively closed judgment sets are the intersections of one or more rational judgments:*

$$\mathcal{J}^+ = \{\cap_{J \in S} J : S \subseteq \mathcal{J}, S \neq \varnothing\}.$$

In particular, each $H \in \mathcal{J}^+$ is the intersection of its rational extensions:

$$H = \cap_{J \in \mathcal{J} : H \subseteq J} J.$$

Proof. First, any intersection $\cap_{J \in S} J$ with $S \subseteq \mathcal{J}$ is deductively closed, and if $S \neq \varnothing$ also consistent, hence in $\mathcal{J}^+$. Conversely, consider any $H \in \mathcal{J}^+$ and define $S = \{J \in \mathcal{J} : H \subseteq J\}$. As H is consistent, $S \neq \varnothing$. We show $H = \cap_{J \in S} J$. Clearly, $H \subseteq \cap_{J \in S} J$. To see why $\cap_{J \in S} J \subseteq H$, note that an $p \in \cap_{J \in S} J$ is entailed by H, hence belongs to H by deductive closure. ■

The following lemma tells us which judgment sets are consistent and deductively closed for our specific agenda:

Lemma 4 *For the special agenda X, the set of consistent and deductively closed judgment sets is*

$$\mathcal{J}^+ = \mathcal{J} \cup \{\{p\} : p \in X\} \cup \{\varnothing\}.$$

Proof. Consider the given agenda. First, $\mathcal{J} \cup \{\{p\} : p \in X\} \cup \{\varnothing\} \subseteq \mathcal{J}^+$, since each rational or singleton or empty judgment set is consistent and moreover deductively closed, for agenda in question. Conversely, consider a judgment set $H \notin \mathcal{J} \cup \{\{p\} : p \in X\} \cup \{\varnothing\}$. We show that $H \notin \mathcal{J}^+$. We can exclude that H contains both members of some issue $\{\pm p_k\}$, as otherwise H is obviously inconsistent, hence outside $\mathcal{J}^+$. As $H \notin \mathcal{J} \cup \{\{p\} : p \in X\} \cup \{\varnothing\}$, the number of issues with which H intersects (defined by $|\{k \in: H \cap \{\pm p_k\} \neq \varnothing\}|$) is not 3, not 1, and not 0. So that number is 2, i.e., H is a two-proposition set. Thus H is not deductively closed, since any two-proposition subset of X entails a third proposition from the remaining issue (e.g., $\{p_1, p_2\}$ entails p_3, and $\{p_1, \neg p_2\}$ entails $\neg p_3$). Hence, $H \notin \mathcal{J}^+$. ■

Proposition 6 *For the special agenda X, every oligarchy is dynamically rational with respect to the above revision operator.*

Proof. Consider the given agenda X and revision operator. Let F be an oligarchy, with set of oligarchs M ($\neq \varnothing$). To prove dynamic rationality, let $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ and $p \in X$. We show that $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$, i.e., that

$$\cap_{i \in M}(J_i|p) = (\cap_{i \in M} J_i)|p.$$

On the left, each $J_i|p$ reduces to $(J_i)_p$ by (11), as $J_i \in \mathcal{J}$. On the right, $(\cap_{i \in M} J_i)|p$ reduces to $\cap_{J \in \mathcal{J}: \cap_{i \in M} J_i \subseteq J} J_p$ by (10), as $\cap_{i \in M} J_i \in \mathcal{J}^+$ by Lemma 3. So we must show that

$$\cap_{i \in M}(J_i)_p = \cap_{J \in \mathcal{J}: \cap_{i \in M} J_i \subseteq J} J_p. \tag{12}$$

By Lemmas 3 and 4, $\cap_{i \in M} J_i$ belongs to $\mathcal{J}^+ = \mathcal{J} \cup \{\{q\} : q \in X\} \cup \{\varnothing\}$. This leads to three cases.

Case 1: $\cap_{i \in M} J_i \in \mathcal{J}$. Here all J_i ($i \in M$) coincide, say with $J^* \in \mathcal{J}$. Hence (12) holds, since both sides equal J^*_p.

Case 2: $\cap_{i \in M} J_i = \{q\}$ for some $q \in X$. Here the sets J_i ($i \in M$) are rational extensions of $\{q\}$, but not all same one extension. So, since $\{q\}$ has just two rational extensions, the sets J_i ($i \in M$) include *all* rational extensions of $\{q\}$; formally, $\{J_i : i \in M\} = \{J \in \mathcal{J} : \{q\} \subseteq J\}$. So each side of (12) equals $\cap_{J \in \mathcal{J}: \{q\} \subseteq J} J_p$, proving (12).

Case 3: $\cap_{i \in M} J_i = \varnothing$. To establish (12), we prove that both sides equal $\{p\}$. The right side of (12) reduces to $\cap_{J \in \mathcal{J}} J_p$, which equals $\{p\}$ by definition of the sets J_p ($J \in \mathcal{J}$). We must show that also the left side equals $\{p\}$. Note that $\cap_{i \in M}(J_i)_p \supseteq \{p\}$, since each $(J_i)_p$ contains p. To prove that $\cap_{i \in M}(J_i)_p \subseteq \{p\}$, we call J' and J'' the two rational extensions of $\{p\}$, and distinguish between three subcases.

Subcase 3.1: J' and J'' are among the sets J_i ($i \in M$). Then

$$\cap_{i \in M}(J_i)_p \subseteq J'_p \cap J''_p = J' \cap J'' = \{p\},$$

where the second equality holds because $J'_p = J'$ (as $p \in J'$) and $J''_p = J''$ (as $p \in J''$). So $\cap_{i \in M}(J_i)_p \subseteq \{p\}$.

Subcase 3.2: Exactly one of J' and J'' is among the sets J_i ($i \in M$). Without loss of generality, let J' but not J'' be among these sets, and let $p = p_1$, $J' = \{p_1, p_2, p_3\}$, and $J'' = \{p_1, \neg p_2, \neg p_3\}$ (other cases are handled analogously). Since $\cap_{i \in M} J_i = \varnothing$, among the sets J_i ($i \in M$) there are sets $\hat{J}$ and $\tilde{J}$ such that $p_2 \notin \hat{J}$ and $p_3 \notin \tilde{J}$. These two sets cannot contain p; otherwise they would be rational extensions of $\{p\}$, so that J' would not be the only rational extension of $\{p\}$ among the sets J_i ($i \in M$). Moreover, $\hat{J} \neq \tilde{J}$; otherwise both sets would equal $\{p_1, \neg p_2, \neg p_3\} = J''$, so that J'' would be among the sets J_i ($i \in M$). These two facts imply that $\hat{J}_p \neq \tilde{J}_p$, by construction of the sets J_p ($J \in \mathcal{J}$). Since $\hat{J}_p$ and $\tilde{J}_p$ are distinct rational extensions of $\{p\}$, $\hat{J}_p \cap \tilde{J}_p = \{p\}$. Meanwhile $\cap_{i \in M}(J_i)_p \subseteq \hat{J}_p \cap \tilde{J}_p$. So $\cap_{i \in M}(J_i)_p \subseteq \{p\}$.

Subcase 3.3: Neither J' nor J'' is among the sets J_i ($i \in M$). This subcase is in fact impossible, because it would imply that all J_i ($i \in M$) are extensions of $\{\neg p\}$, contradicting that $\cap_{i \in M} J_i = \varnothing$. ■

B.5 Possibilities without systematicity

We have defined a non-systematic escape route, in the form of a particular asymmetric unanimity rule. To make the rule dynamically rational, we have assumed a special non-simple agenda X and an equally special revision operator. We now establish that the revision operator has the desired properties, and that the rule is indeed dynamically rational with respect to it; the rule obviously satisfies all other conditions of Theorem 2 except systematicity, in fact except the neutrality part of systematicity.

Proposition 7 *For the given non-simple agenda, the specified revision operator is regular and rationality-preserving.*

Proof. Assume this agenda X. Revision is obviously regular. To see that revision preserves rationality, consider any rational J and any $p \in X$. $J|p$ is complete because Y contains a member of each pair $\{\pm q\} \subseteq X$. $J|p$ is consistent because it includes neither Y, nor any pair $\{\pm q\}$, hence includes no minimal inconsistent set. ■

Proposition 8 *For the given non-simple agenda, the specified asymmetric unanimity rule is dynamically rational with respect to the given revision operator.*

Proof. Consider the given agenda, revision operator, and aggregation rule. To verify dynamic rationality, consider any $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ and any $p \in X$ such that $\langle J_1|p, ..., J_n|p \rangle \in \mathcal{J}^n$ (in fact, membership of $\langle J_1|p, ..., J_n|p \rangle$ in $\mathcal{J}^n$ already follows from the fact that X contains no contradictory proposition and revision preserves rationality). We distinguish between three cases:

- *Case 1:* $p \in Y$ and $p \in J_1, ..., J_n$. Then $p \in F(J_1, ..., J_n)$. By conservativeness, neither any of $J_1, ..., J_n$ nor $F(J_1, ..., J_n)$ changes by learning p. So $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$.
- *Case 2:* $p \in Y$ and $p \notin J_i$ for some i. Since all of $J_1|p, ..., J_n|p$ contain p by successfulness, and since $J_i|p$ contains $\neg y$ for all $y \in Y \backslash \{p\}$, we have $F(J_1|p, ..., J_n|p) = \{p\} \cup \{\neg y : y \in Y \backslash \{p\}\}$. Meanwhile, as $p \in Y$ and $p \notin J_i$, we have $p \notin F(J_1, ..., J_n)$, so that $F(J_1, ..., J_n)|p = \{p\} \cup \{\neg y : y \in Y \backslash \{p\}\}$. So, $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$.
- *Case 3:* $p \notin Y$. Then the revised profile $\langle J_1|p, ..., J_n|p \rangle$ displays unanimous acceptance of p and (as $p \notin Y$) coincides with the initial profile $\langle J_1, ..., J_n \rangle$ outside the issue $\{\pm p\}$. So, $F(J_1|p, ..., J_n|p)$ contains p and coincides with $F(J_1, ..., J_n)$ outside $\{\pm p\}$. Also $F(J_1, ..., J_n)|p$ contains p and (because $p \notin Y$) coincides with $F(J_1, ..., J_n)$ outside $\{\pm p\}$. Hence, $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$. ■

B.6 Possibilities for simple agendas

If the agenda is simple, then plenty of aggregation rules satisfy the conditions of the theorem, in the case of dynamic rationality assuming a particular revision operator defined above. This result was stated as 'Proposition 1'. We now show first that this revision operator satisfies our desiderata, and then that Proposition 1 holds. Notation is as above.

Proposition 9 *The specified revision operator is regular, and for a simple agenda also rationality-preserving.*

Proof. Consider the specified revision operator. Let $J \subseteq X$ and $p \in J$. If $J \notin \mathcal{J}$, then regularity applied to J and p holds by stipulation, and rationality preservation applied to J and p holds vacuously. Now suppose $J \in \mathcal{J}$. We must show three things.

1. (*successfulness*) We have to show that $p \in J|p$. Note that J contains p or $\neg p$, as $J \in \mathcal{J}$. In the first case, $J|p$ contains p_p, which equals p because $\{p, p\}$ is consistent (as $J \in \mathcal{J}$). In the second case, $J|p$ contains $(\neg p)_p$, which equals p as $\{\neg p, p\}$ is inconsistent. So, in any case, $p \in J|p$.

2. (*conservativeness*) If $p \in J$, then $J|p = J$ because for each $q \in J$ we have $q_p = q$ (since $\{q, p\}$ is consistent, being included in the rational judgment set J).

3. (*rationality preservation*) Assume X is simple. For a contradiction, let p be non-contradictory and let $J|p \notin \mathcal{J}$. Then J is inconsistent, hence has a minimal inconsistent subset Y. By simplicity of X, $|Y| \leq 2$, say $Y = \{q_p, q'_p\}$ for some $q, q' \in J$. By

definition of q_p and q'_p (and by p's non-contradictoriness), the sets $\{q_p, p\}$ and $\{q'_p, p\}$ are consistent. Since $\{q_p, p\}$ is consistent and q_p entails $\neg q'_p$, also $\{\neg q'_p, p\}$ is consistent. Similarly, since $\{q'_p, p\}$ is consistent and q'_p entails $\neg q_p$, $\{\neg q_p, p\}$ is consistent. Now, as $\{q_p, p\}$ and $\{\neg q_p, p\}$ are consistent, $q_p = q$ by definition. Analogously, as $\{q'_p, p\}$ and $\{\neg q'_p, p\}$ are consistent, $q'_p = q'$. So, $\{q_p, q'_p\} = \{q, q'\}$, a subset of the consistent set J. This contradicts the inconsistency of $\{q_p, q'_p\}$. ■

Proof of Proposition 1. Consider a simple agenda X, a unanimity-preserving independent rule $F : \mathcal{J}^n \to \mathcal{J}$, and the revision operator defined above. To prove dynamic rationality, consider a profile $\langle J_1, ..., J_n\rangle \in \mathcal{J}^n$ and a learnt proposition $p \in X$ such that $\langle J_1|p, ..., J_n|p\rangle \in \mathcal{J}^n$ (i.e., such that p is non-contradictory). To prove that $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$, we fix a $q \in X$ and show that $F(J_1|p, ..., J_n|p)$ and $F(J_1, ..., J_n)|p$ coincide on q. We distinguish between three cases.

- *Case 1*: $\{q, p\}$ and $\{\neg q, p\}$ are both consistent. Then $J|p$ coincides with J on q for all $J \in \mathcal{J}$. In particular, $J_i|p$ coincides with J_i on q for all i, and hence (by independence) $F(J_1|p, ..., J_n|p)$ coincides with $F(J_1, ..., J_n)$ on q. Meanwhile $F(J_1, ..., J_n)|p$ also coincides with $F(J_1, ..., J_n)$ on q. So, $F(J_1|p, ..., J_n|p)$ coincides with $F(J_1, ..., J_n)|p$ on q.
- *Case 2*: $\{q, p\}$ is consistent and $\{\neg q, p\}$ is inconsistent. Then $q \in J|p$ for all $J \in \mathcal{J}$. In particular, $q \in J_i|p$ for all i, whence by unanimity preservation and independence $q \in F(J_1|p, ..., J_n|p)$. Meanwhile $q \in F(J_1, ..., J_n)|p$. So $F(J_1|p, ..., J_n|p)$ and $F(J_1, ..., J_n)|p$ coincide on q.
- *Case 3*: $\{q, p\}$ is inconsistent and $\{\neg q, p\}$ is consistent. Then $\neg q \in J|p$ for all $J \in \mathcal{J}$. In particular, $\neg q \in J_i|p$ for all i, whence by unanimity preservation and independence $\neg q \in F(J_1|p, ..., J_n|p)$. Meanwhile $\neg q \in F(J_1, ..., J_n)|p$. So $F(J_1|p, ..., J_n|p)$ and $F(J_1, ..., J_n)|p$ coincide on $\neg q$, hence on q. ■

C Proof of both possibility theorems

We now prove both possibility theorems about premise-based aggregation and revision. All definitions and notation apply. In particular, recall that

- any premise subagenda X_{prem} induces a conclusion subagenda $X_{\text{conc}} = X \backslash X_{\text{prem}}$, a set of premise issues $\mathcal{Z}_{\text{prem}} \subseteq \mathcal{Z}$, and a set of conclusion issues $\mathcal{Z}_{\text{conc}} \subseteq \mathcal{Z}$,
- any premise subagenda X_{prem}, premise aggregators $(F_Z)_{Z \in \mathcal{Z}_{\text{prem}}}$, and consequence rule Cn jointly induce a premise-based rule F on $\widehat{\mathcal{J}}^n$,
- any premise subagenda X_{prem}, premise revisors $(|_Z)_{Z \in \mathcal{Z}_{\text{prem}}}$, and consequence rule Cn jointly induce a premise-based revision operator.

C.1 Theorem 3

To prove Theorem 3, fix a proper (premise) subagenda X_{prem}, a consequence rule Cn, and an idempotent premise-based revision operator. Let $F : \widehat{\mathcal{J}}^n \to \widehat{\mathcal{J}}$ be a premise-based rule with unanimity-preserving premise aggregators $F_Z : \mathcal{J}_Z^n \to \mathcal{J}_Z$ ($Z \in \mathcal{Z}_{\text{prem}}$). To show that F is dynamically rational, consider any profile $\langle J_1, ..., J_n \rangle \in \widehat{\mathcal{J}}^n$ and learnt proposition $p \in X$ such that $\langle J_1|p, ..., J_n|p \rangle \in \widehat{\mathcal{J}}^n$. We must show that $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$. This is done by proving that, for all issues $Z \in \mathcal{Z}$,

$$F(J_1|p, ..., J_n|p) \cap Z = [F(J_1, ..., J_n)|p] \cap Z. \tag{13}$$

Claim 1: Equation (13) holds for all premise issues $Z \in \mathcal{Z}_{\text{prem}}$.

Consider any $Z \in \mathcal{Z}_{\text{prem}}$. For each individual i, let q_i be the single member of $J_i \cap Z$. Also, let q_0 be the single member of $F_Z(J_1 \cap Z, ..., J_n \cap Z)$. Then,

$$F_Z(\{q_1\}, ..., \{q_n\}) = \{q_0\}. \tag{14}$$

The left side of the desired equation (13) is rewritable as follows:

$$\begin{aligned} F(J_1|p, ..., J_n|p) \cap Z &= F_Z((J_1|p) \cap Z, ..., (J_n|p) \cap Z) \\ &= F_Z(\{q_1\}|^Z p, ..., \{q_n\}|^Z p), \end{aligned}$$

where the first and second equation holds by definition of premise-based aggregation and revision, respectively. Meanwhile the right side of the desired equation equals $\{q_0\}|^Z p$, by definition of premise-based revision. So the desired equation reduces to

$$F_Z(\{q_1\}|^Z p, ..., \{q_n\}|^Z p) = \{q_0\}|^Z p. \tag{15}$$

In other words, we must show that F_Z is (in the obvious sense) dynamically rational at the local profile $(\{q_1\}, ..., \{q_n\})$ and the learnt proposition p. There are two cases:

- *Case 1*: learning p does not lead to revision of any judgments on Z, i.e., $\{q\}|^Z p = \{q\}$ for each $\{q\} \in \mathcal{J}_Z$. Then the desired equation (15) reduces to the known equation (14), hence is true.
- *Case 2*: learning p leads to revision of some judgment on Z, i.e., there is a $\{q\} \in \mathcal{J}_Z$ such that $\{q\}|^Z p \neq \{q\}$.
  - *Subcase 2.1*: $\{q\}|^Z p \in \mathcal{J}_Z$. Here, as $\{q\}|^Z p \neq \{q\}$, we must have $\{q\}|^Z p = \{\neg q\}$, and thus, by idempotence of revision, $\{\neg q\}|^Z p = \{\neg q\}$. So the desired equation (15) reduces to

    $$F_Z(\{\neg q\}, ..., \{\neg q\}) = \{\neg q\},$$

    which holds because F_Z preserves unanimity.

– *Subcase 2.2*: $\{q\}|^Z p \notin \mathcal{J}_Z$. Recall that, by assumption, for each individual i we have $J_i|p \in \widehat{\mathcal{J}}$; hence $\{q_i\}|^Z p \in \mathcal{J}_Z$, which (because $\{q\}|^Z p \notin \mathcal{J}_Z$) implies that $q_i \neq q$. So, $q_i = \neg q$. Hence, by (14), $F_Z(\{\neg q\}, ..., \{\neg q\}) = \{q_0\}$. By unanimity preservation it follows that $q_0 = \neg q$. Hence the desired equation (15) reduces to
$$F_Z(\{\neg q\}|^Z p, ..., \{\neg q\}|^Z p) = \{\neg q\}|^Z p,$$
which holds because F_Z preserves unanimity. (In fact, $\{\neg q\}|^Z p$ must equal $\{\neg q\}$; otherwise $\{\neg q\}|^Z p$ would equal $\{q\}$, whence $(\{\neg q\}|^Z p)|^Z p = \{q\}|^Z p \neq \{\neg q\}$, contradicting idempotence.)

Claim 2: Equation (13) holds for all conclusion issues $Z \in \mathcal{Z}_{\text{conc}}$.

Consider any $Z \in \mathcal{Z}_{\text{conc}}$. By definition of premise-based aggregation, the left side of the desired equation (15) equals

$$Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} [F(J_1|p, ..., J_n|p) \cap Z']) \cap Z,$$

while by definition of premise-based revision the right side of the desired equation equals

$$Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} [F(J_1, ..., J_n)|p \cap Z']) \cap Z.$$

So the desired equation becomes

$$Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} [F(J_1|p, ..., J_n|p) \cap Z']) \cap Z = Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} [F(J_1, ..., J_n)|p \cap Z']) \cap Z.$$

This holds by Claim 1. ■

C.2 Theorem 4

To prove Theorem 4, we consider a premise-based revision operator, given by a premise subagenda X_{prem}, a consequence rule Cn, and premise revisors $(|_Z)_{Z \in \mathcal{Z}_{\text{prem}}}$. We assume revision is idempotent, and regular on premises.

Part 1. First consider a premise-based rule F. It obviously maps from $\widehat{\mathcal{J}}^n$ to $\widehat{\mathcal{J}}$. By Theorem 3, it is dynamically rational, provided its premise aggregators F_Z preserve unanimity. It is independent of irrelevant propositions, because the collective judgment on a proposition $p \in X$ is entirely fixed by the individual judgments on the *relevant* proposition (in $\mathcal{R}(p)$), whether p is a premise or a conclusion, as is clear from the definition or premise-based rules. Finally, provided each F_Z is monotonic, F is globally monotonic, by the following argument: if in a profile in $\widehat{\mathcal{J}}^n$ one replaces someone's judgment set by the collective judgment set, then that individual's judgment on each premise issue $Z \in \mathcal{Z}_{\text{prem}}$ is replaced by the collective judgment on Z, which by monotonicity of each

F_Z ($Z \in \mathcal{Z}_{\text{prem}}$) has no effect on collective judgments on premises, and thus has no effect on conclusions either since aggregation is premise-based.

Part 2. Conversely, assume $F : \widehat{\mathcal{J}}^n \to \widehat{\mathcal{J}}$ is a dynamically rational aggregation rule that is independent of irrelevant propositions and globally monotonic. Let G be the premise-based rule whose premise aggregators $F_Z : \mathcal{J}_Z^n \to \mathcal{J}_Z$ ($Z \in \mathcal{Z}_{\text{prem}}$) are defined as follows. For any premise issue $Z \in \mathcal{Z}_{\text{prem}}$ and any local profile $(L_1, ..., L_n) \in \mathcal{J}_Z^n$, let $F_Z(L_1, ..., L_n) = F(J_1, ..., J_n) \cap Z$ for some (hence, by independence of irrelevant propositions, *any*) profile $\langle J_1, ..., J_n \rangle \in \widehat{\mathcal{J}}^n$ such that $L_1 \subseteq J_1, ..., L_n \subseteq J_n$. Since F maps into $\widehat{\mathcal{J}}$, each F_Z indeed maps into $\mathcal{J}_Z$.

We must prove that $F = G$ and that each F_Z is unanimity-preserving and monotonic.

Claim 1. $F = G$.

Fix a profile $\langle J_1, ..., J_n \rangle \in \widehat{\mathcal{J}}^n$ and write $J_F := F(J_1, ..., J_n)$ and $J_G := G(J_1, ..., J_n)$. We prove that $J_F = J_G$ by showing that, for all issues $Z \in \mathcal{Z}$,

$$J_F \cap Z = J_G \cap Z. \tag{16}$$

Firstly, equation (16) holds for all premise issues $Z \in \mathcal{Z}_{\text{prem}}$, because each side then equals $F_Z(J_1 \cap Z, ..., J_n \cap Z)$. Now fix a conclusion issues $Z \in \mathcal{Z}_{\text{conc}}$. By definition of premise-based rules,

$$J_G \cap Z = Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} (J_G \cap Z')) \cap Z. \tag{17}$$

Turning to J_F, and using repeatedly that F is globally monotonic,

$$\begin{aligned} J_F &= F(J_1, ..., J_n) \\ &= F(J_F, J_2, ..., J_n) \\ &= F(J_F, J_F, J_3, ..., J_n) \\ &\cdots \\ &= F(J_F, ..., J_F), \end{aligned}$$

where we have used in each step that the new profile still lies in the domain of F since $J_F \in \widehat{\mathcal{J}}$. Now pick any $p \in J_F \cap X_{\text{prem}}$ (noting that $J_F \cap X_{\text{prem}} \neq \varnothing$ because $J_F \in \widehat{\mathcal{J}}$). Now $J_F \cap X_{\text{prem}} = J_F|p \cap X_{\text{prem}}$, since revision is conservative on premises. So, since F is independent of irrelevant propositions (and only premises are relevant to any propositions),

$$F(J_F, ..., J_F) = F(J_F|p, ..., J_F|p) = F(J_F, ..., J_F)|p.$$

where the second equality holds by dynamic rationality. Therefore, since $J_F = F(J_F, ..., J_F)$, we have shown that $J_F = J_F|p$. Meanwhile, since revision is premise-based,

$$(J_F|p) \cap Z = Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}}((J_F|p) \cap Z') \cap Z.$$

Replacing $J_F|p$ by J_F, we obtain

$$J_F \cap Z = Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}}(J_F \cap Z') \cap Z. \tag{18}$$

By (17), (18), and Claim 1, we can deduce (16). Q.e.d.

Claim 2. Each premise aggregator F_Z ($Z \in \mathcal{Z}_{\text{prem}}$) preserves unanimity.

Consider any $Z \in \mathcal{Z}_{\text{prem}}$ and any unanimous local profile $(L, ..., L) \in \mathcal{J}_Z^n$, say $L = \{p\}$. We must show that $F_Z(L, ..., L) = L$, or equivalently (as $F_Z(L, ..., L)$ belongs to $\mathcal{J}_Z$ and is thus singleton) that $p \in F_Z(L, ..., L)$. Choose any extension $J \supseteq L$ in $\widehat{\mathcal{J}}$. Since revision is conservative on premises, $(J|p) \cap X_{\text{prem}} = J \cap X_{\text{prem}}$. Hence, not just J, but also $J|p$ is a member of $\widehat{\mathcal{J}}$ that extends L. So, $F_Z(L, ..., L)$ equals $F(J|p, ..., J|p) \cap Z$, which equals $F(J, ..., J)|p$ by dynamic rationality. As revision is successful on premises, p belongs to $F(J, ..., J)|p$, hence to $F_Z(L, ..., L)$. Q.e.d.

Claim 3. Each premise aggregator F_Z ($Z \in \mathcal{Z}_{\text{prem}}$) is monotonic.

The argument is simple. Consider any F_Z ($Z \in \mathcal{Z}_{\text{prem}}$) and any local profile $(L_1, ..., L_n) \in \mathcal{J}_Z^n$. Note that ordinary and global monotonicity are equivalent given the local nature of the agenda (and the fact that F_Z maps into the same set $\mathcal{J}_Z$ to which also individual judgment sets belong, so that we can substitute collective for individual judgment sets without leaving the domain of F_Z). So let us show global monotonicity of F_Z. Let $(L_1, ..., L, ..., L_n)$ arise from $(L_1, ..., L_n)$ by replacing some individual i's judgment set L_i by $L = F_Z(L_1, ..., L_n)$. We must show that $F(L_1, ..., L, ..., L_n) = L$. Pick extensions $J_1 \supseteq L_1, ..., J_n \supseteq L_n$ in $\widehat{\mathcal{J}}$. Define $J = F(J_1, ..., J_n)$. Note that $L = J \cap Z$, and so $L \subseteq J$. Now

$$\begin{aligned}
L &= F_Z(L_1, ..., L_n) \\
&= F(J_1, ..., J_n) \cap Z \text{ as } L_1 \subseteq J_1, ..., L_n \subseteq J_n \\
&= F(J_1, ..., J, ..., J_n) \cap Z \text{ as } F \text{ is monotonic} \\
&= F_Z(L_1, ..., L, ..., L_n) \text{ as } L_1 \subseteq J_1, ..., L \subseteq J, ..., L_n \subseteq J_n. \blacksquare
\end{aligned}$$